A Syntactic Characterization of the Gabbay–de Jongh Logics

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$\dashv A$ means that A is refutable

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there is a counter model to A

Łukasiewicz (1951)

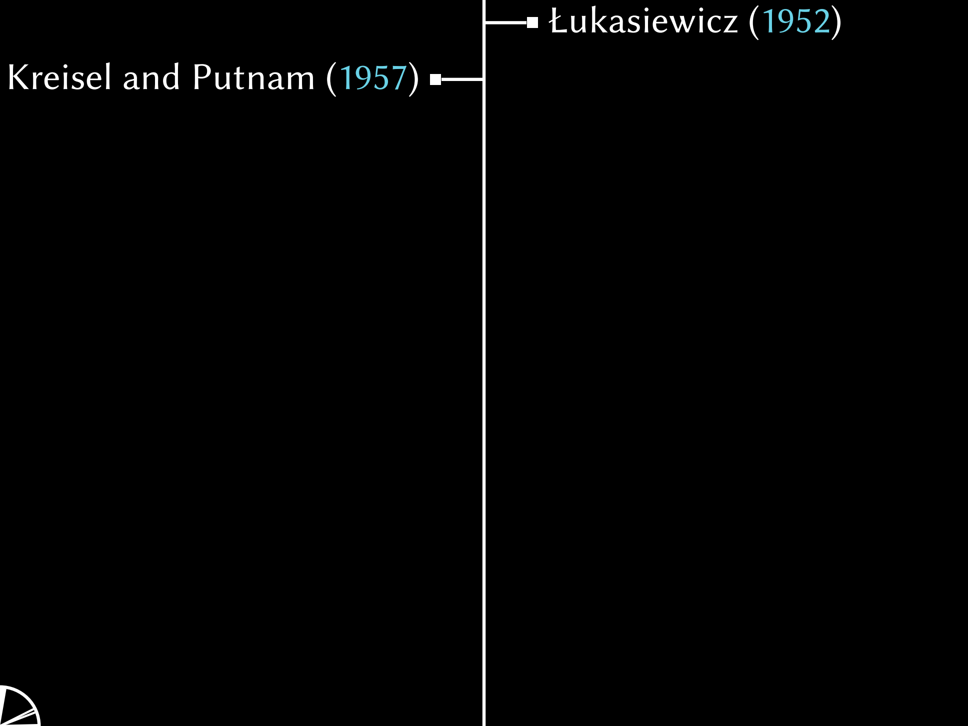
$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

Łukasiewicz (1952)

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

$$\frac{\dashv A \qquad \dashv B}{\dashv A \vee B}$$

Łukasiewicz (1952)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Kreisel and Putnam (1957)

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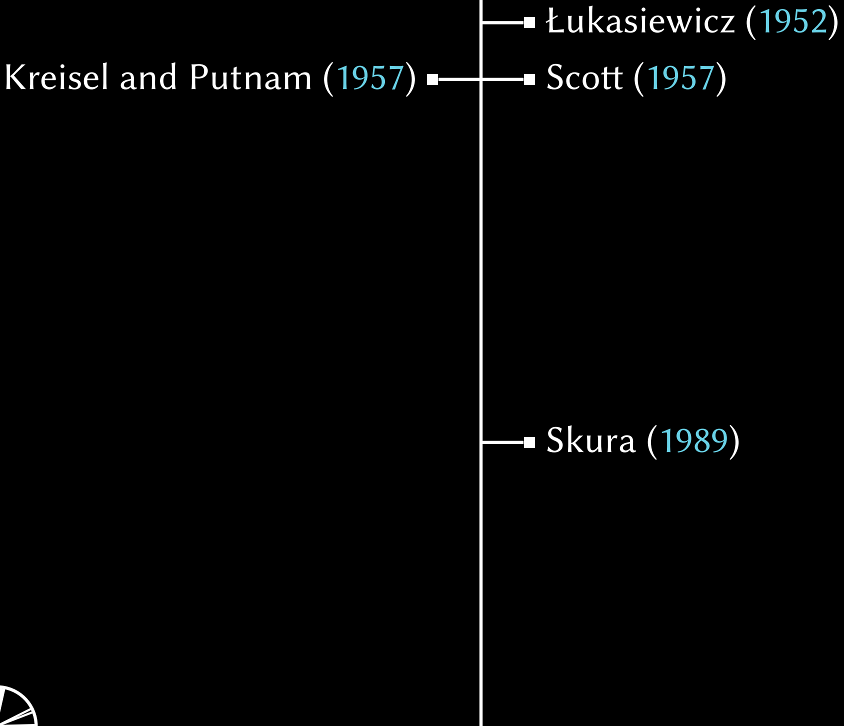
Kreisel and Putnam (1957)

Scott (1957)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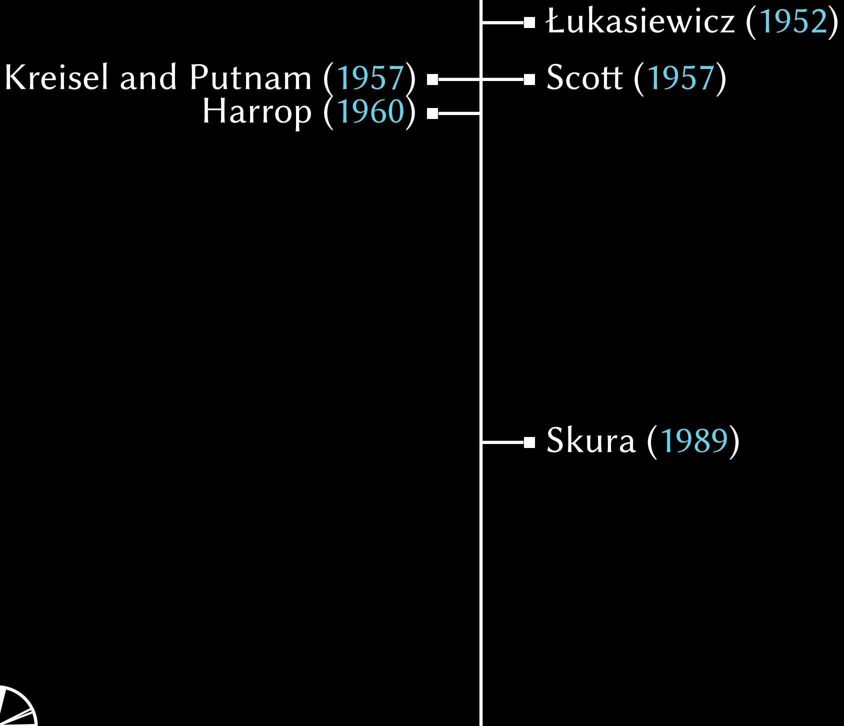
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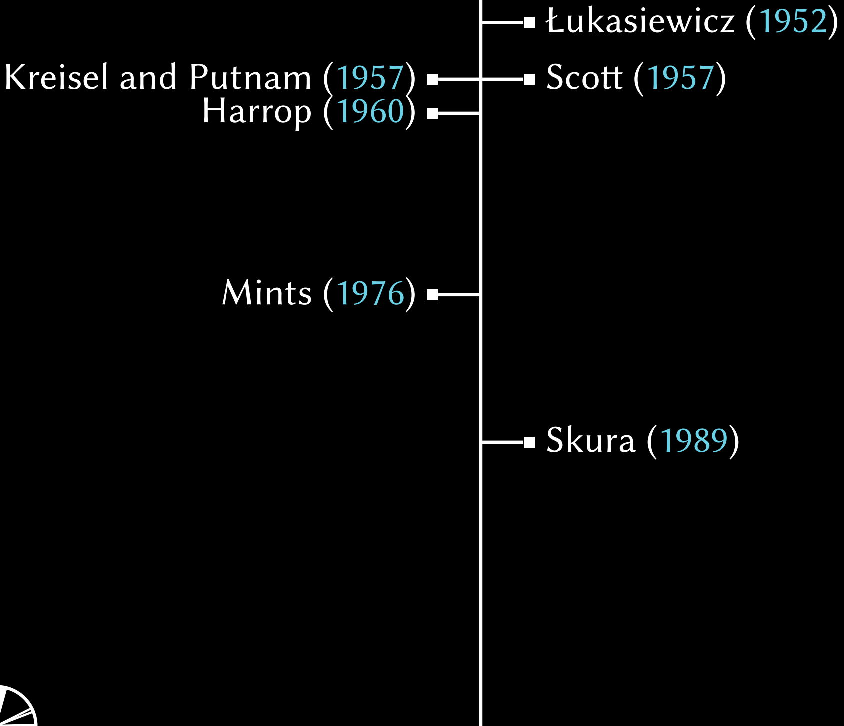
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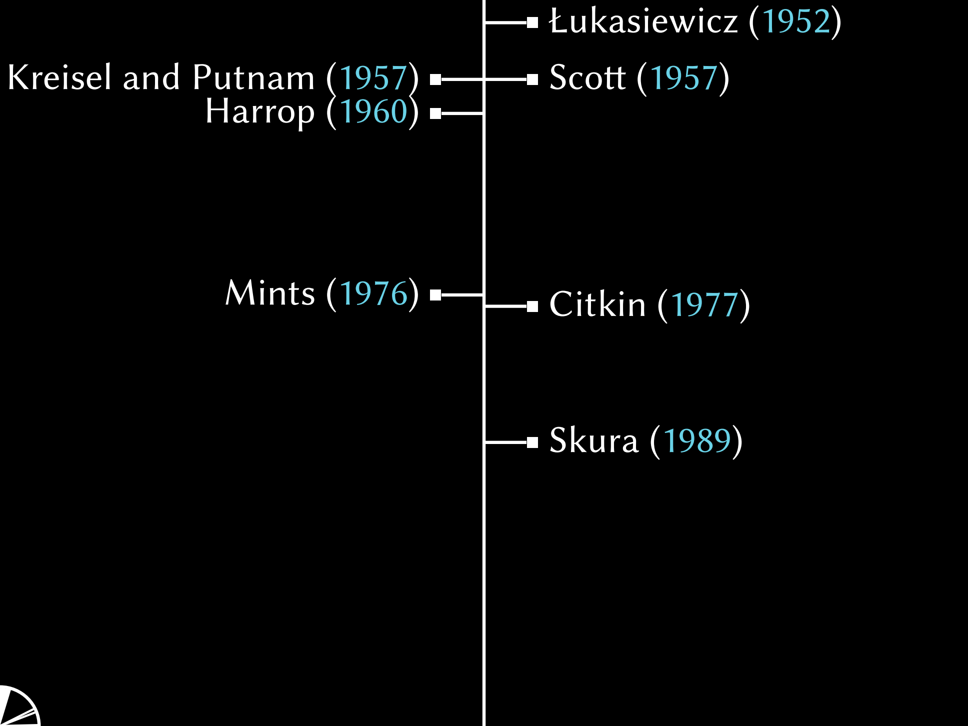
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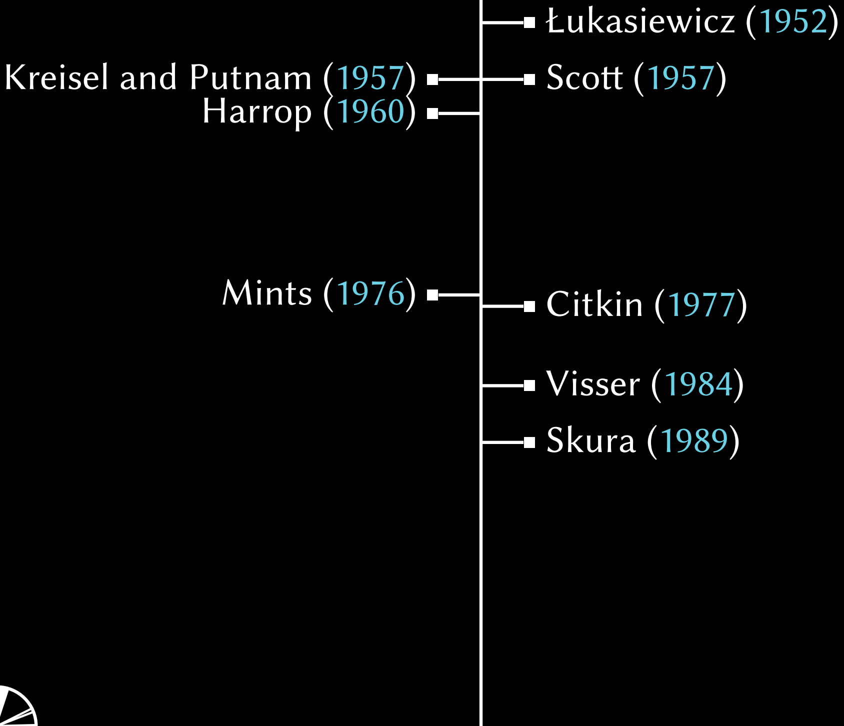
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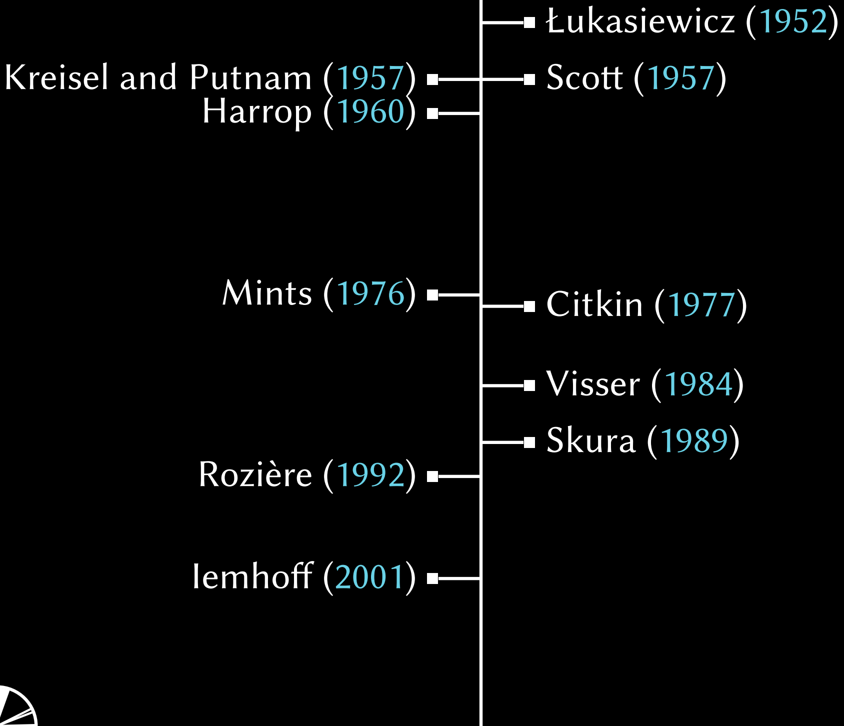
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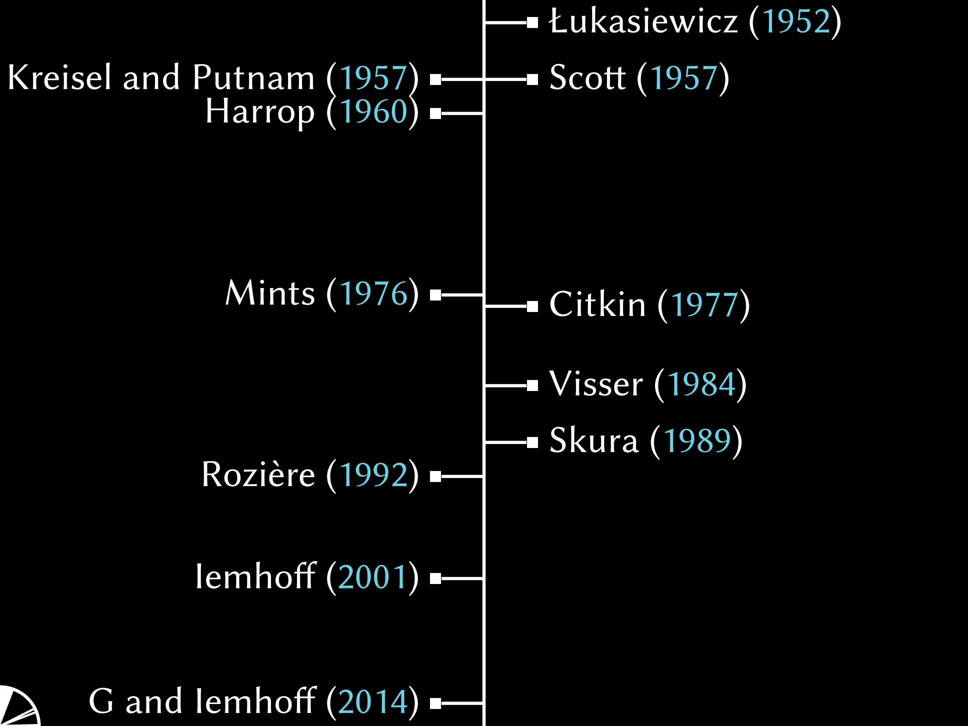
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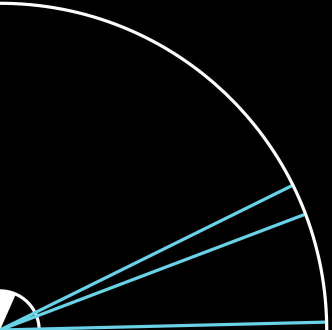
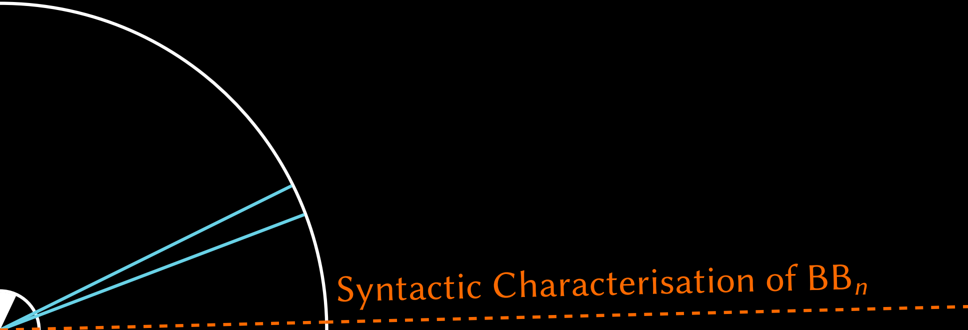
Iemhoff (2001)
G and Iemhoff (2014)

Overview

Overview



Overview



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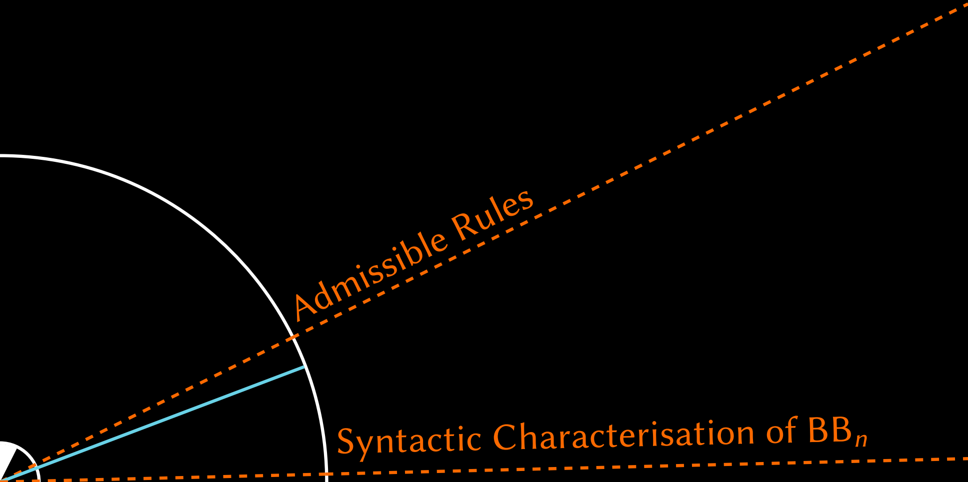

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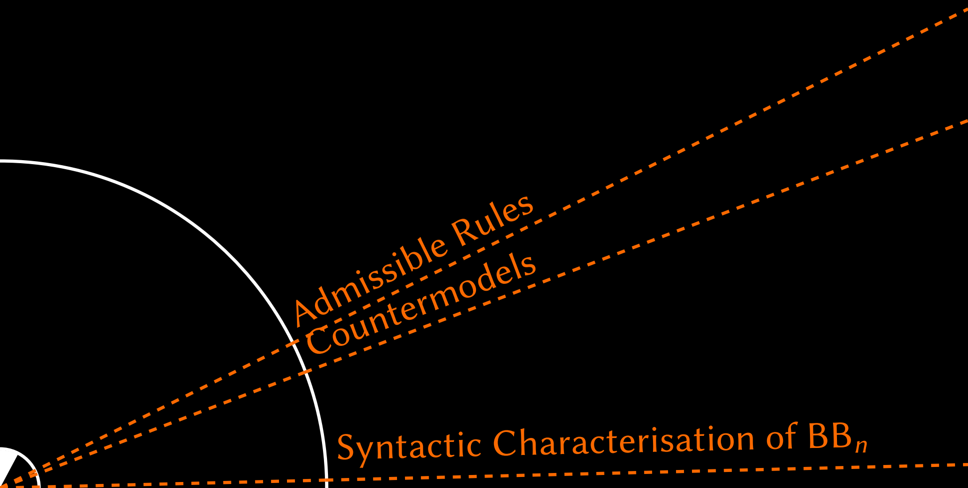


Admissible Rules
Countermodels
Syntactic Characterisation of BB_n

Admissible Rules

A / $\triangle$ admissible

σA is derivable

$A \;/\; \Delta$ admissible

σC is derivable for some $C \in \Delta$

σA is derivable

$$A \mathrel{|\!\!\sim} \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

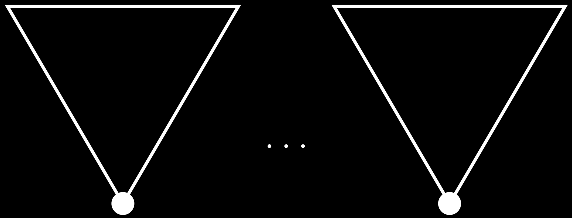
Admissible Rules

$$\frac{\bigvee \Delta}{\{C \mid C \in \Delta\}}$$

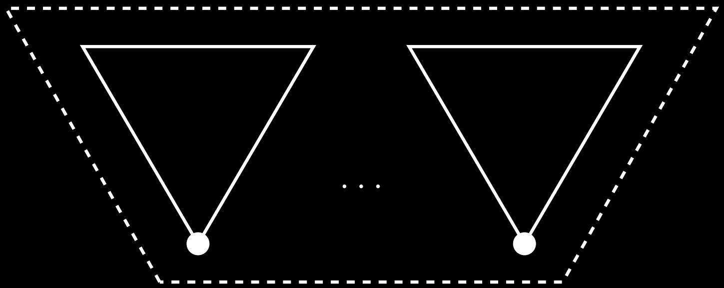
Admissible Rules

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\{\, (\bigvee \Delta \to A) \to C \mid C \in \Delta \}}$$

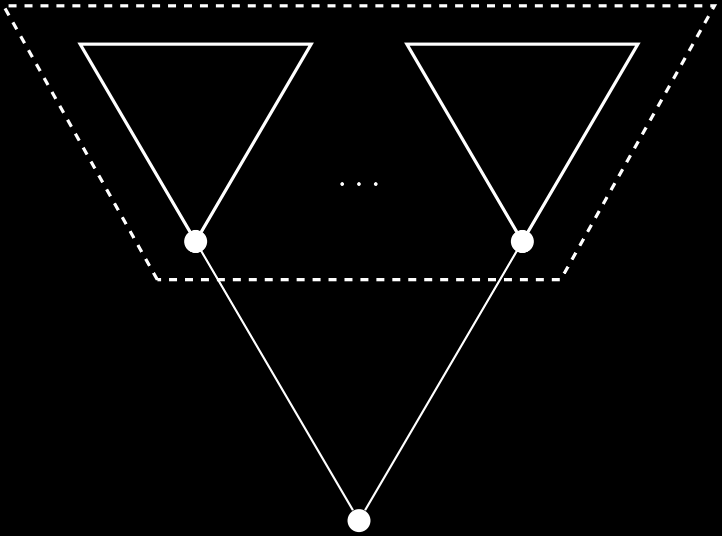
n^{th} Extension Property



n^{th} Extension Property



n^{th} Extension Property



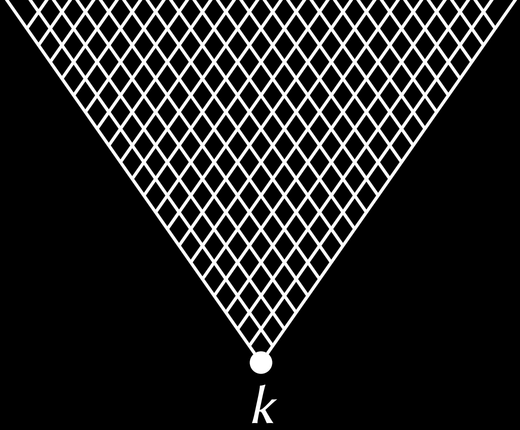
Jankov–de Jongh formulae

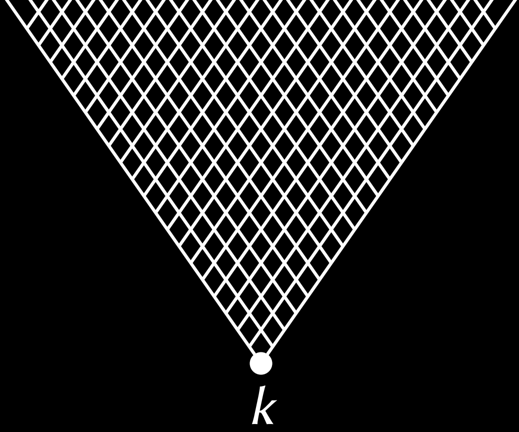
In suitable models have

$$l \Vdash \mathrm{up}\, k \quad \text{iff} \quad k \leq l$$

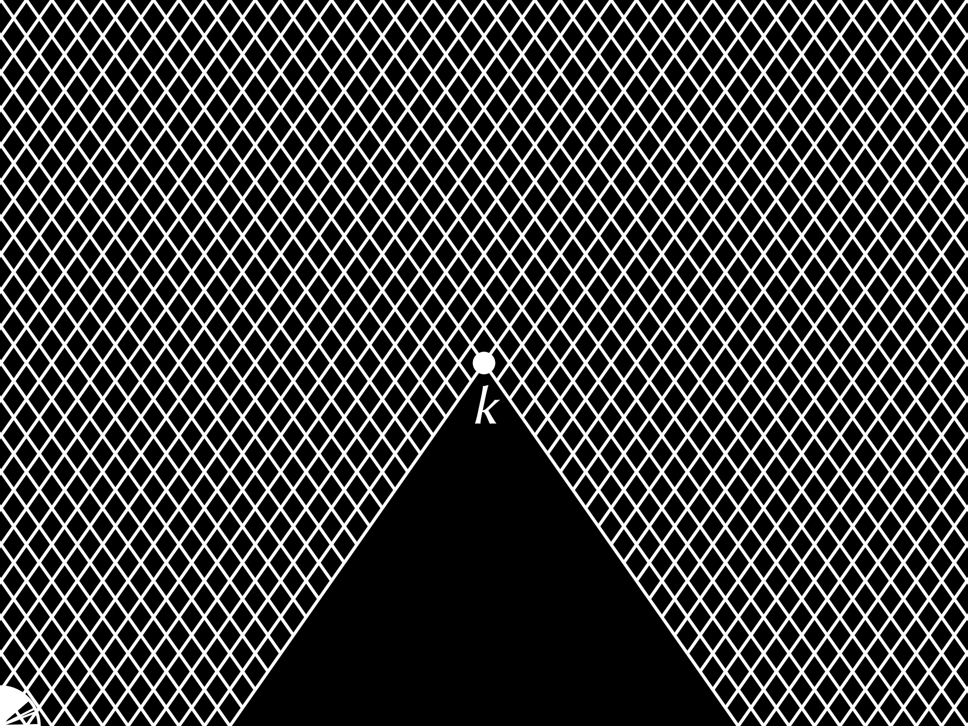
$$l \Vdash \mathrm{nd}\, k \quad \text{iff} \quad l \not\leq k$$

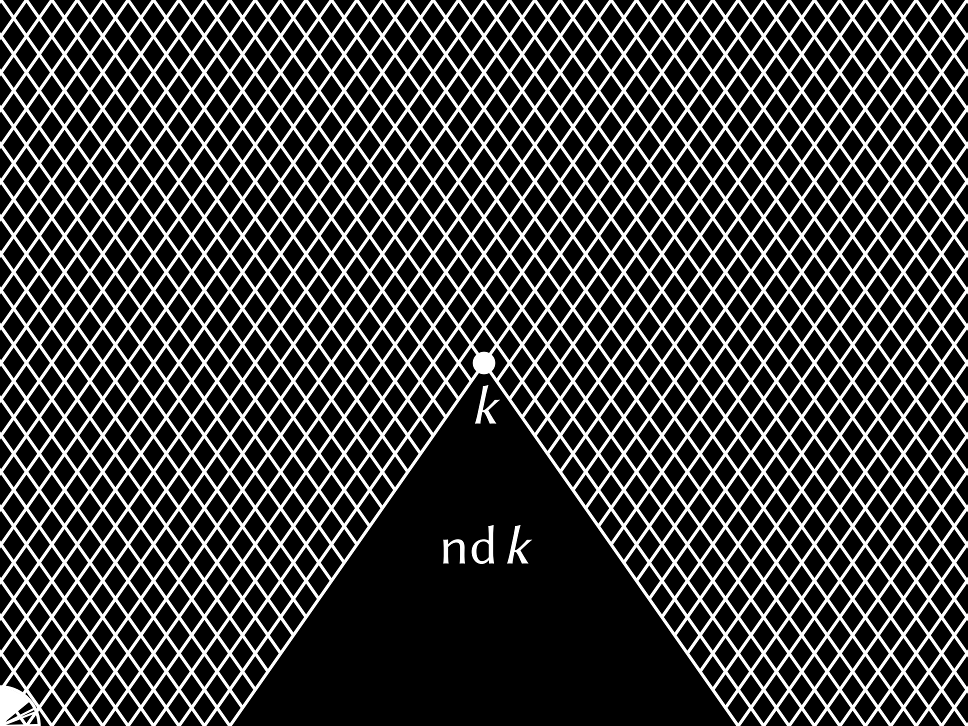

k


k



up k

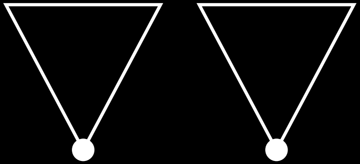

k

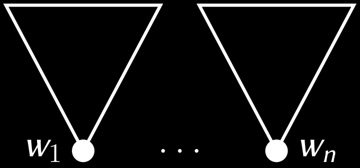

k
nd k

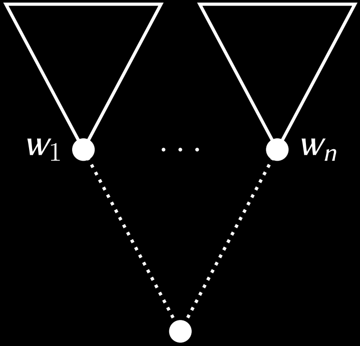
Per X there is a suitable model which “contains a copy” of each finite rooted model on X.

Per *X* there is a suitable model which "contains a copy" of each finite rooted model on *X*.

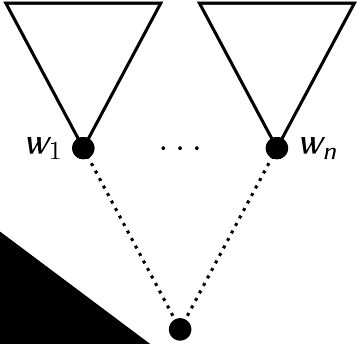
Bellissima (1986) · Rybakov (1994)
Shehtman (1978) · Grigolia (1987)

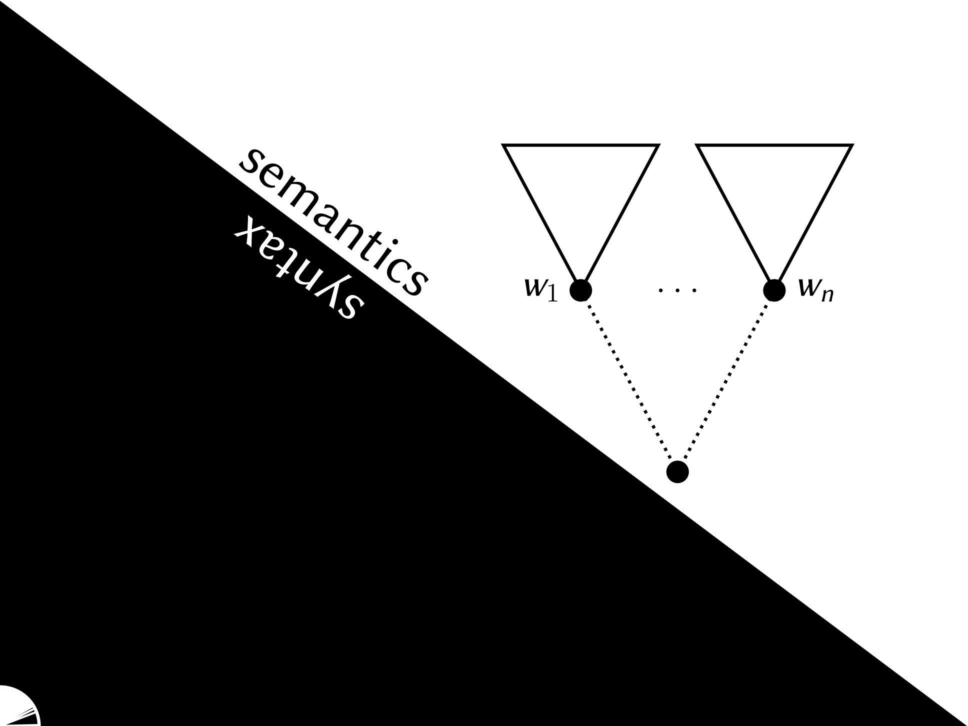



w_1
. . .
w_n


w_1
$\cdots$
w_n

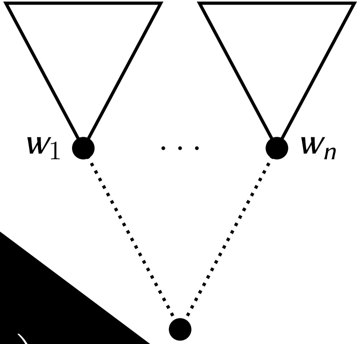
semantics




semantics
syntax
w_1
. . .
w_n

semantics

syntax

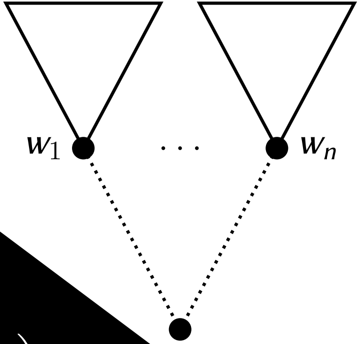


$$\vdash \bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i \right) \to \mathsf{nd}\, w_j$$

$$\text{or} \nvdash \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i \right) \to \bigvee_{i=1}^{n} \mathsf{nd}\, w_i$$

semantics

syntax



$$\vdash \bigvee_{C \in \Delta} \left(\bigvee \Delta \quad \to A \qquad\qquad \right) \to C$$

$$\text{or} \nvdash \left(\bigvee \Delta \quad \to A \qquad\qquad \right) \to \bigvee \Delta$$

A logic with DP and FMP
admits the Visser rules up to *n*
iff it has
the extension property up to *n*.

Countermodels

$\vdash$ is **admissibly reducible** to Θ

whenever

$\nvdash A$ iff $\sigma A \mid\!\sim \Delta$

for some σ and $\Delta \subseteq \Theta$

if $\vdash$ is admissibly reducible to Θ

then $\nvdash \, = \, \dashv$.

$$\frac{D \in \Theta}{\dashv D} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{\dashv C \text{ per } C \in \Delta \quad A \mathrel{|\!\!\sim} \Delta}{\dashv A}$$

Gabbay–de Jongh Logics

Logic of Bounded Branching

$$\mathrm{BB}_n = \mathrm{IPC} + \bigwedge_{i=0}^{n} \left(\left(x_i \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{i=0}^{n} x_i$$

$BB_n \not\vdash A$ iff there is a proper & at most n-fold branching tree T with $T \not\Vdash A$.

A poset K is

maximally separable

whenever

$$a \leq b$$

for all maximal $m \in K$

$b \leq m$ implies $a \leq m$

A tree is proper
iff it is
maximally separable.

$$
\begin{aligned}
\mathrm{m}_K : \; K &\to \mathrm{P}(K) \\
k &\mapsto \{k\} \text{ if } k \text{ max else } \emptyset
\end{aligned}
$$

If $K \not\Vdash A$ then

$\sigma A \vdash \mathrm{nd}\, K$ for some σ.

If K is at most n-fold branching then

$$\mathrm{nd}\,K \vdash\!\sim \left\{ \mathrm{nd}\,m \;\middle|\; m \in K \text{ maximal} \right\}$$

Characterisation

$\mathrm{BB}_n \nvdash A$

$$\mathrm{BB}_n \nvdash A$$

$$T \nVdash A$$

$$\mathrm{BB}_n \nvdash A$$

$$T \nVdash A$$

$$\sigma A \vdash \mathrm{nd}\, T$$

$$\mathrm{BB}_n \nvdash A$$
$$T \nVdash A$$

$$\sigma A \vdash \mathrm{nd}\,T$$
$$\mathrel{|\!\!\sim} \{\mathrm{nd}\,m \mid m \in T \text{ maximal}\}$$

BB_n reduces admissibly to CPC-non-derivable formulae

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

$$\frac{\dashv (\bigvee_{i=1}^{n} C_i \to A) \to C_j \text{ for all } j}{\dashv (\bigvee_{i=1}^{n} C_i \to A) \to \bigvee_{j=1}^{n} C_j}$$

BB_n is the intermediate logic where

$$\nvdash \, = \, \dashv$$

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