A Syntactic Characterization of the Gabbay–de Jongh Logics

Jeroen Goudsmit
Utrecht University
July 31st 2013

$\dashv A$ means that A is refutable

$\dashv A$ means that A is refutable

there is a counter model to A

Łukasiewicz (1951)

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

$\dashv A$

$$\frac{\dashv \sigma A}{\dashv A}$$

$$\frac{\dfrac{\dashv x \qquad \sigma A \vdash x}{\dashv \sigma A}}{\dashv A}$$

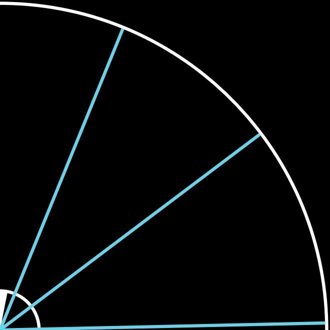
Łukasiewicz (1952)

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

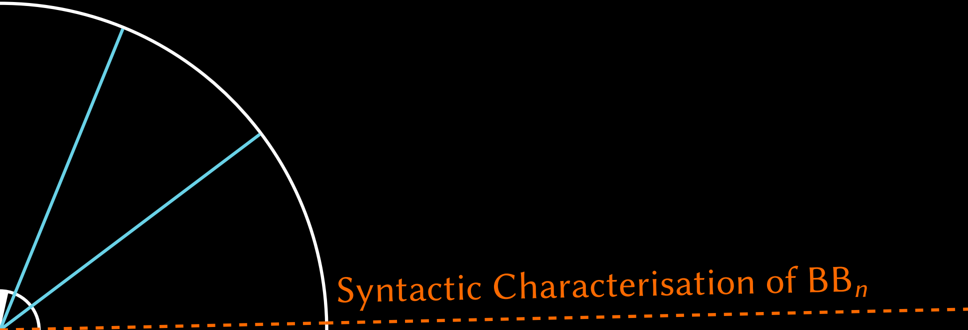
$$\frac{\dashv A \qquad \dashv B}{\dashv A \vee B}$$

Overview

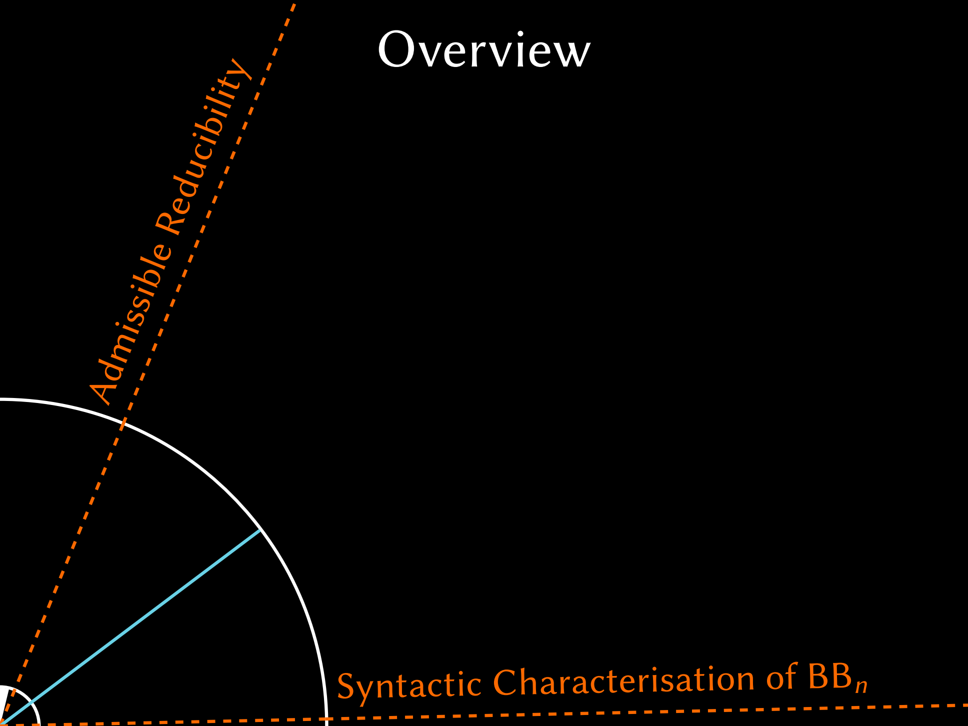
Overview



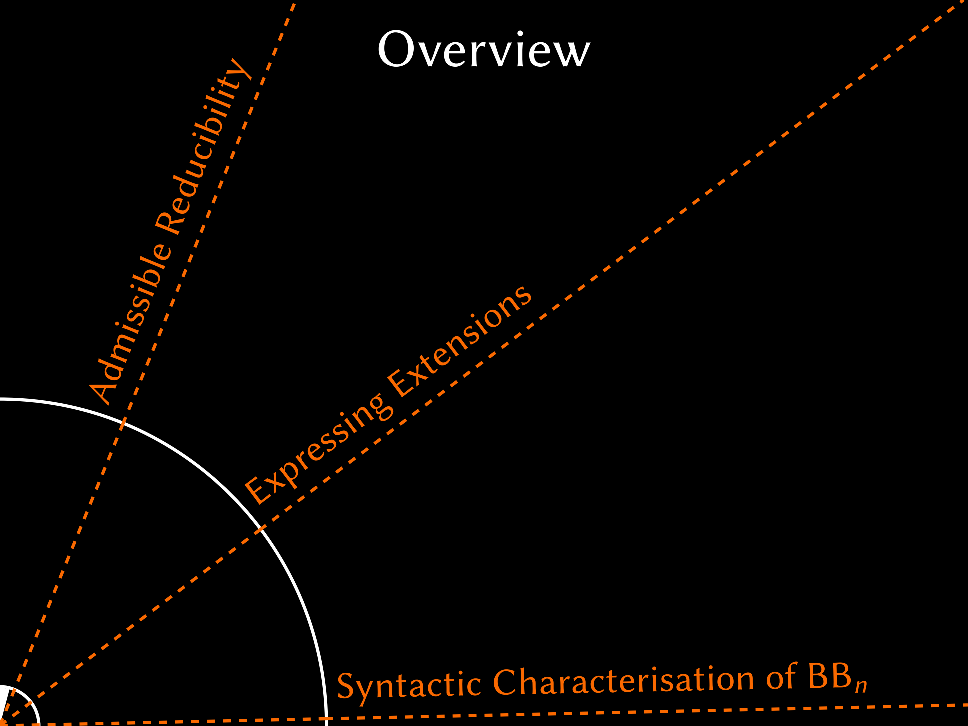
Overview



Overview


Admissible Reducibility
Syntactic Characterisation of BB$_n$

Overview



Admissible Reducibility
Expressing Extensions
Syntactic Characterisation of BB_n

A / $\triangle$ admissible

σA is derivable

$$A \;/\; \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

σA is derivable

$A \mathrel{|\!\sim} \Delta$ admissible

σC is derivable for some $C \in \Delta$

$$\frac{\neg C \to A \vee B}{(\neg C \to A) \vee (\neg C \to B)}$$

$$\frac{\neg C \rightarrow A \vee B}{\{\ \neg C \rightarrow A, \qquad \neg C \rightarrow B\ \}}$$

$\vdash$ is admissibly reducible to Θ

whenever

$\nvdash A$ iff $\sigma A \mathrel{|\!\sim} \Theta$ for some σ

if $\vdash$ is admissibly reducible to Θ
then $\nvdash \; = \; \dashv$.

$$\frac{D \in \Theta}{\dashv D} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{\dashv C \text{ per } C \in \Delta \quad A \mathrel{|\!\sim} \Delta}{\dashv A}$$

Gabbay–de Jongh Logics

Logic of Bounded Branching

$$\mathrm{BB}_n = \mathrm{IPC} + \bigwedge_{i=0}^{n} \left(\left(x_i \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{i=0}^{n} x_i$$

$BB_n \nvdash A$ iff there is a proper & at most n-fold branching tree T with $T \nVdash A$.

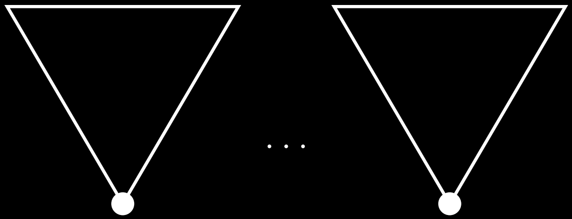
Visser Rules

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\bigvee \{(\bigvee \Delta \to A) \to C \mid C \in \Delta\}}$$

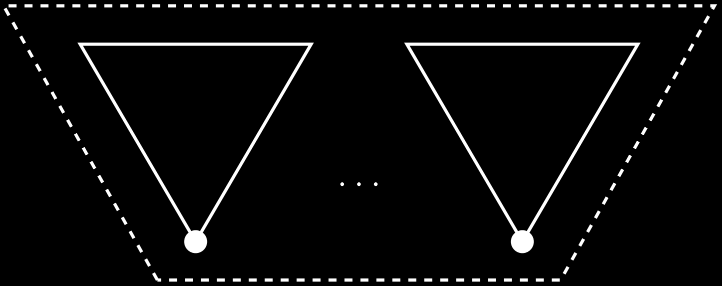
Visser Rules

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\{(\bigvee \Delta \to A) \to C \mid C \in \Delta\}}$$

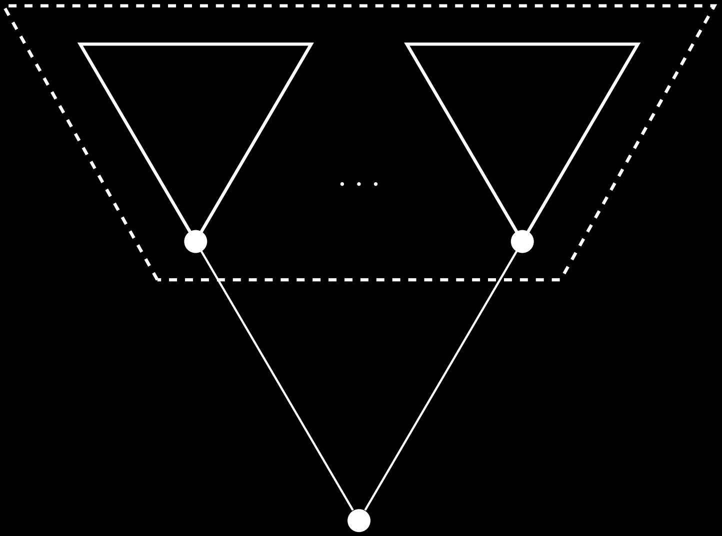
Extension Property



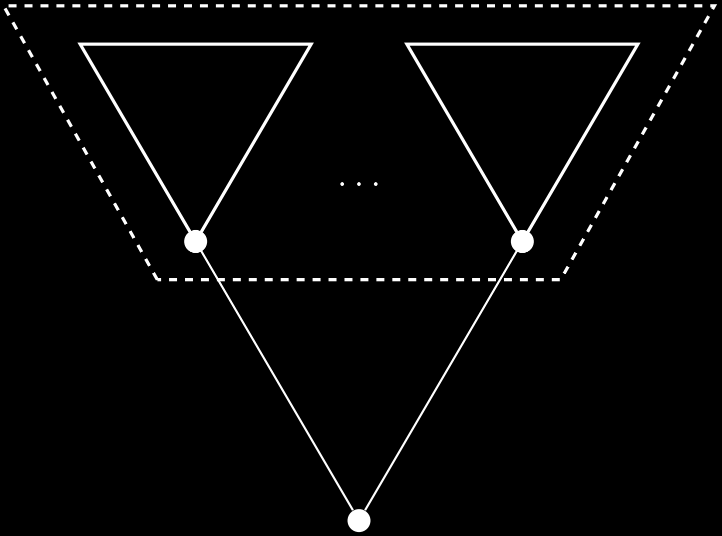
Extension Property



Extension Property



n^{th} Extension Property

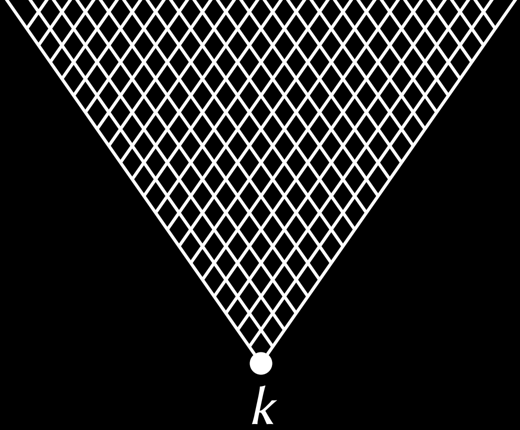


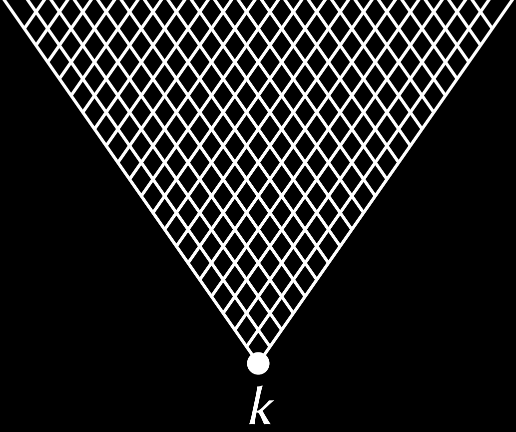
Jankov–de Jongh formulae

In suitable models have

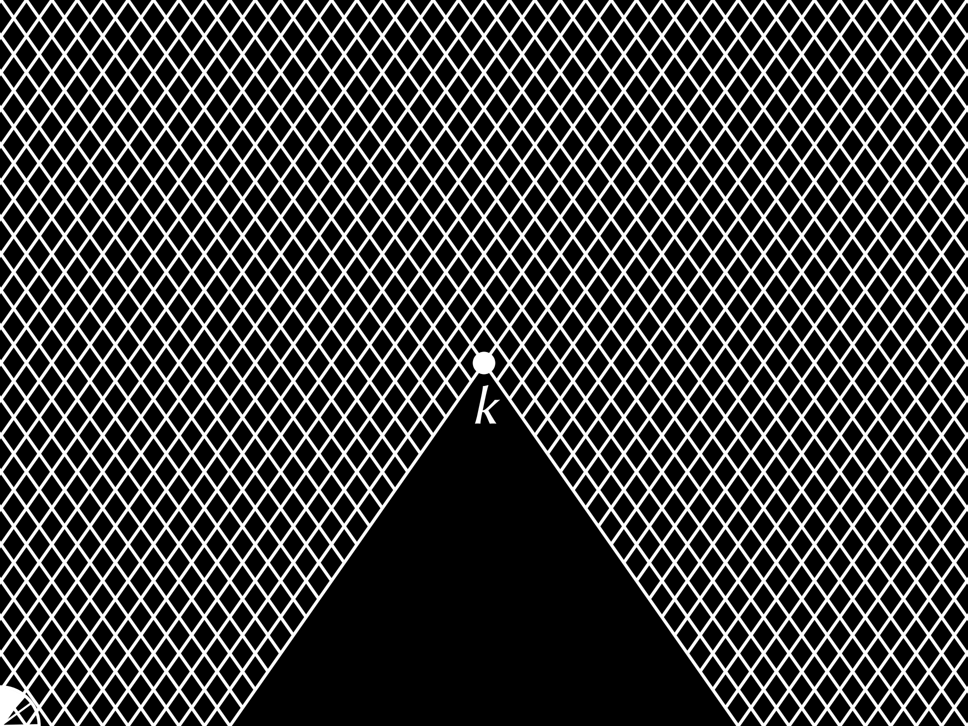
$$l \Vdash \mathrm{up}\, k \quad \text{iff} \quad k \leq l$$
$$l \Vdash \mathrm{nd}\, k \quad \text{iff} \quad l \nleq k$$

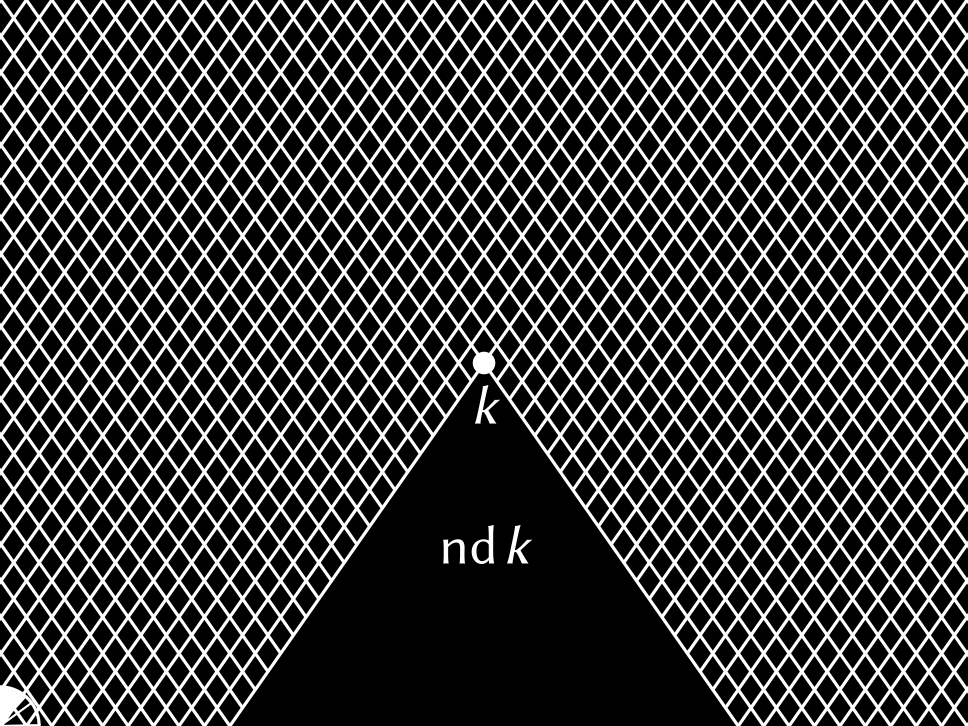

k


k



up *k*


k

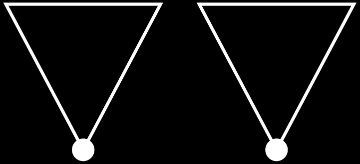


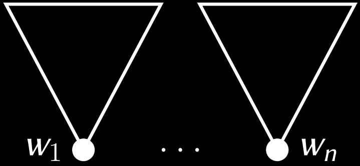
k
nd k

Per X there is a suitable model which “contains a copy” of each finite model on X.

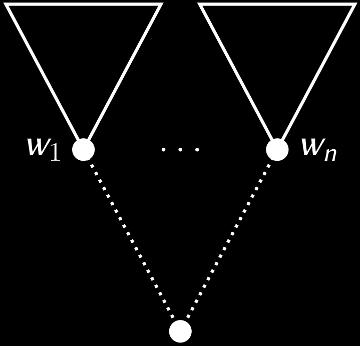
Per X there is a suitable model which "contains a copy" of each finite model on X.

Bellissima (1986) · Rybakov (1994)
Shehtman (1978) · Grigolia (1987)



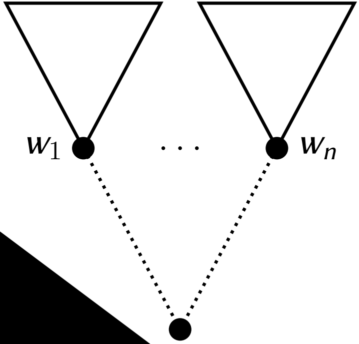


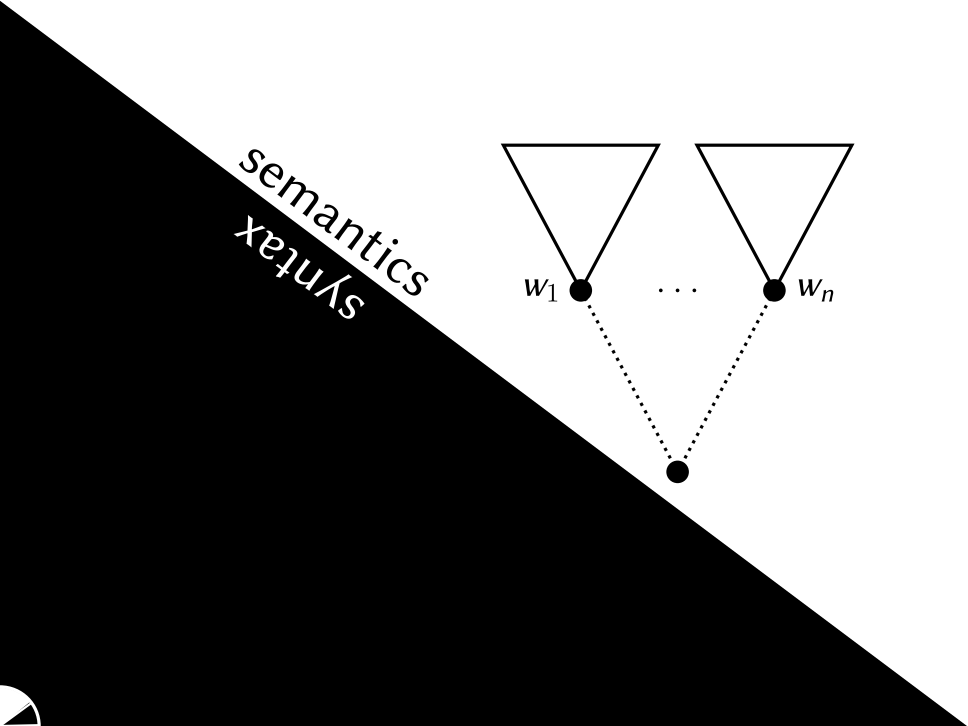
w_1
. . .
w_n



w_1
$\cdots$
w_n

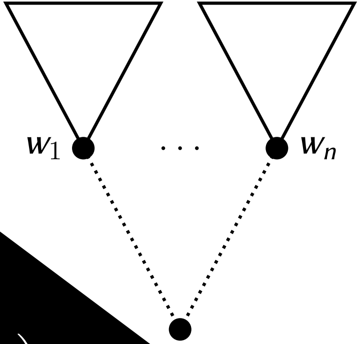
semantics




semantics
syntax
w_1
. . .
w_n

semantics

syntax

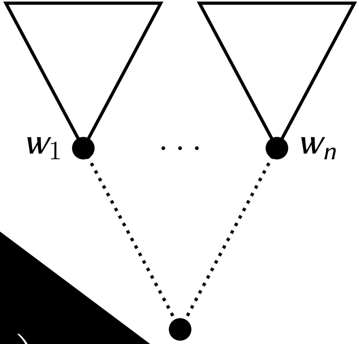


$$\vdash \bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} \mathrm{nd}\, w_i \to \bigvee\nolimits_{i=1}^{n} \mathrm{up}\, w_i \right) \to \mathrm{nd}\, w_j$$

$$\text{or } \nvdash \left(\bigvee_{i=1}^{n} \mathrm{nd}\, w_i \to \bigvee\nolimits_{i=1}^{n} \mathrm{up}\, w_i \right) \to \bigvee_{i=1}^{n} \mathrm{nd}\, w_i$$

semantics

syntax

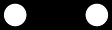


$$\vdash \bigvee_{C \in \Delta} \left(\bigvee \Delta \quad \to A \qquad \right) \to C$$

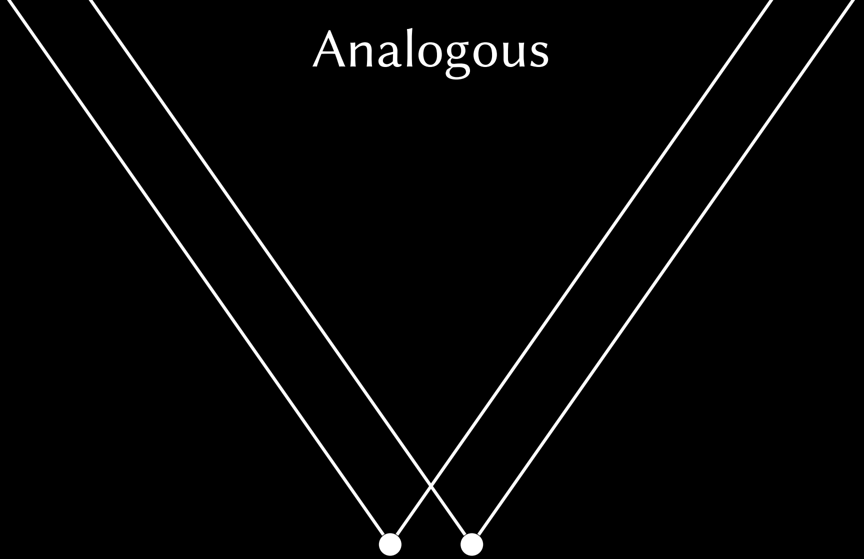
$$\text{or } \nvdash \left(\bigvee \Delta \quad \to A \qquad \right) \to \bigvee \Delta$$

A logic with DP and FMP
admits the Visser rules up to n
iff it has
the extension property up to n.

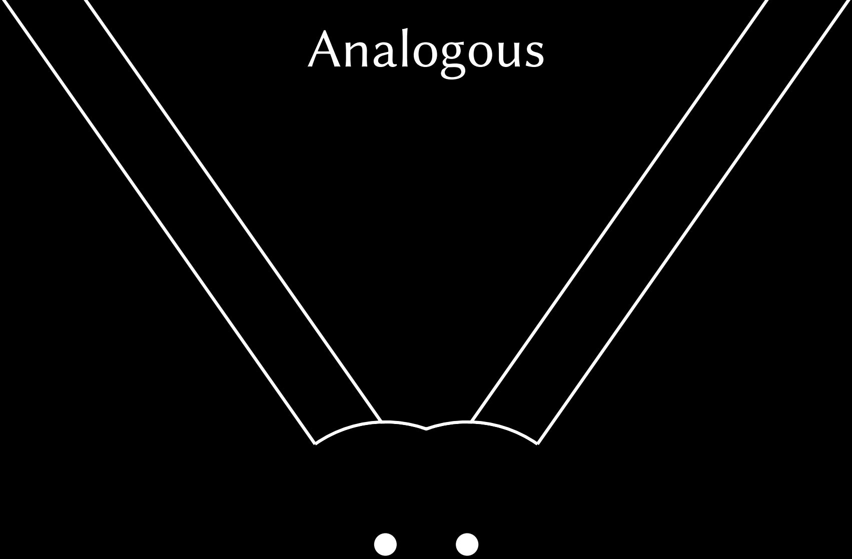
Analogous



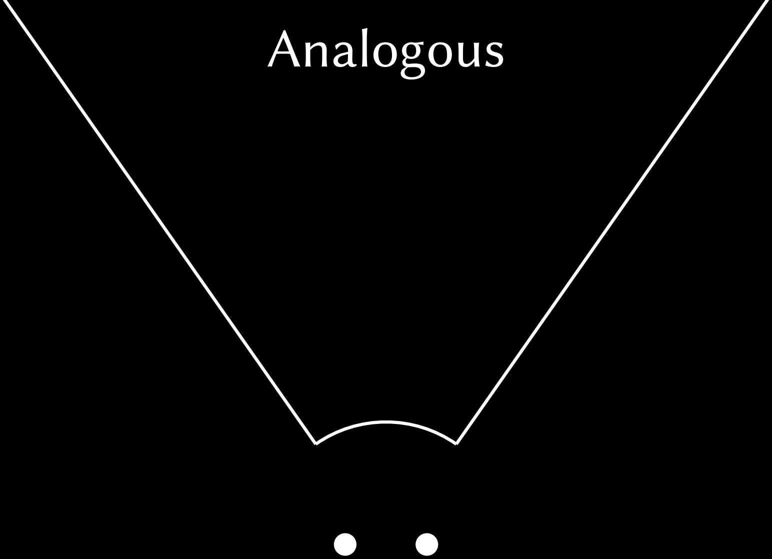
Analogous



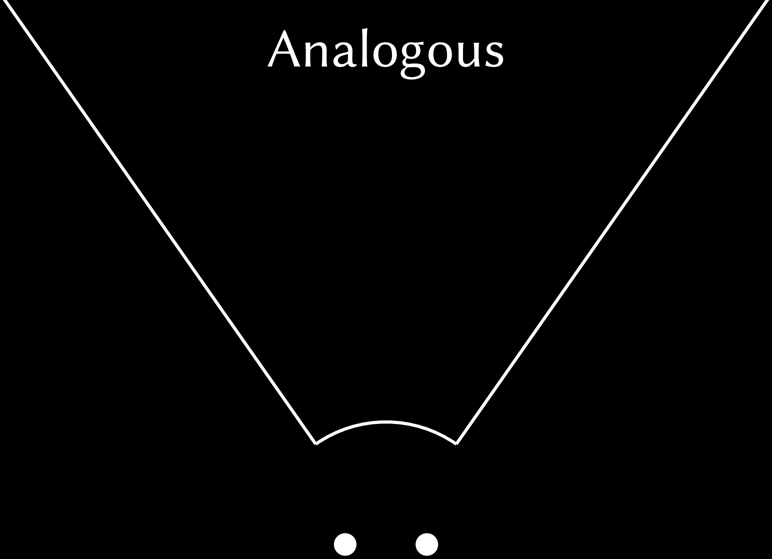
Analogous



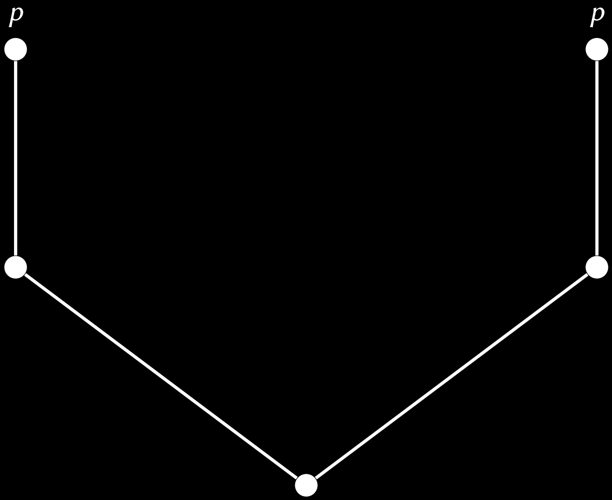
Analogous

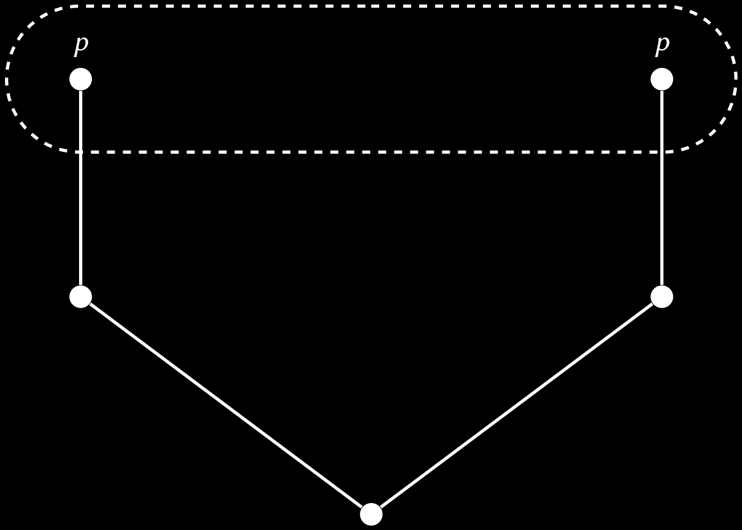


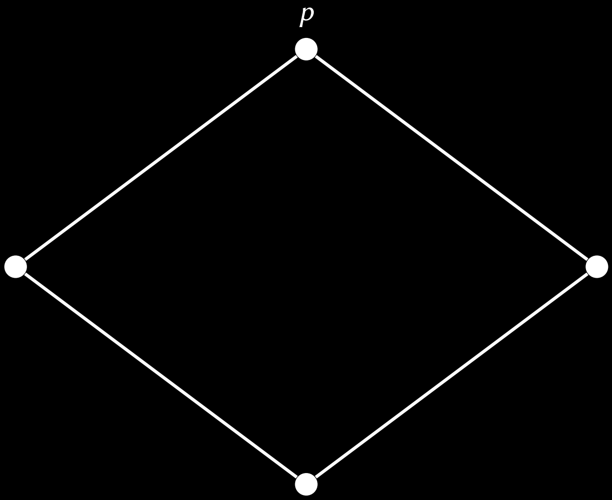
Analogous

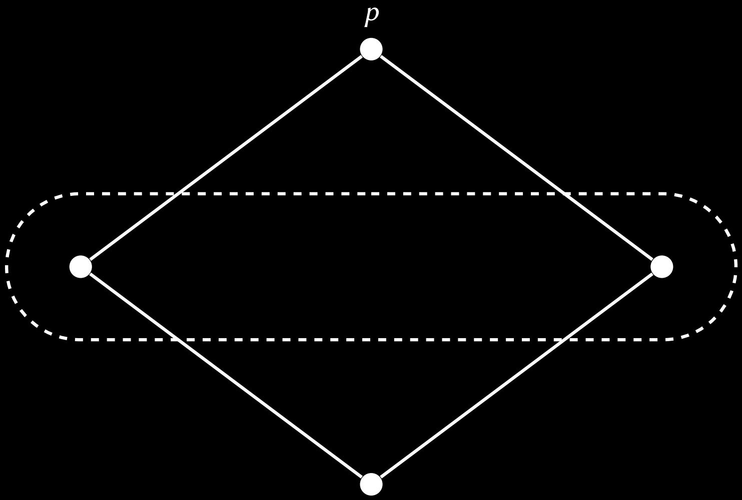


$k \equiv l$ when $v(k) = v(l)$ and $k \leq u$ iff $l \leq u$ for all $u \neq k, l$

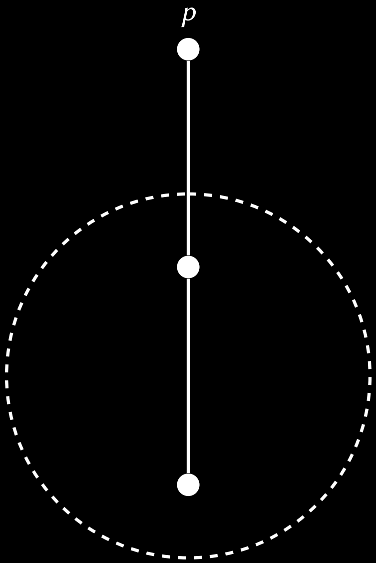

p
p


p
p


p


p


p


p


p

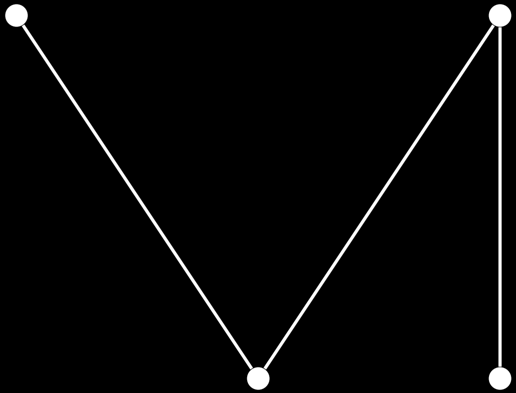
$$t_T : T \to \mathrm{P}\,T$$

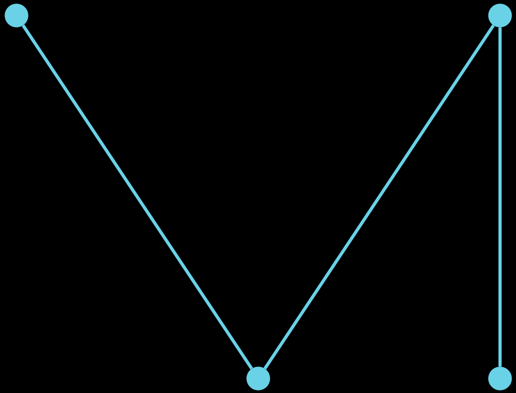
$$t_T, k \Vdash l \text{ iff } l = k \text{ and } k \text{ is maximal}$$

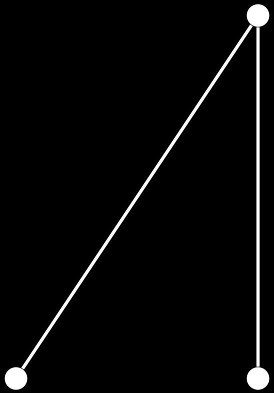
A poset T is called
maximally distinguishable
if in the model t_T
analogous is equal.

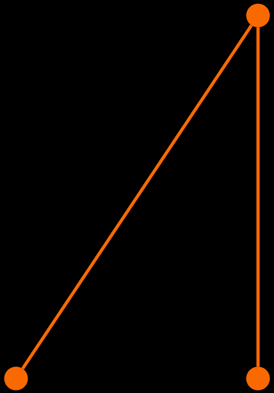
$$t_T : T \to \mathrm{P}\,T$$

$$t_T, k \Vdash l \text{ iff } l = k \text{ and } k \text{ is maximal}$$









Every proper tree is maximally distinguishable.

If T is proper & maximally distinguishable then $\mathrm{nd}\, t_T \mathrel{|\!\!\sim} \{\mathrm{nd}\, w \mid w \in T \text{ maximal}\}$.

$$\mathrm{BB}_n \nvdash A$$

$$\mathrm{BB}_n \nvdash A$$
$$T \nVdash A$$

$$\mathrm{BB}_n \nvdash A$$

$$T \nVdash A$$

$$\sigma A \vdash \mathrm{nd}\, t_T$$

$$\begin{aligned} \mathrm{BB}_n &\nvdash A \\ T &\nVdash A \end{aligned}$$

$$\begin{aligned} \sigma A &\vdash \mathrm{nd}\, t_T \\ &\mathrel{|\!\sim} \{\mathrm{nd}\, w \mid w \in T \text{ maximal}\} \end{aligned}$$

BB_n reduces admissibly to CPC-non-derivable formulae

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

$$\frac{\dashv (\bigvee_{i=1}^{n} C_i \to A) \to C_j \text{ for all } j}{\dashv (\bigvee_{i=1}^{n} C_i \to A) \to \bigvee_{j=1}^{n} C_j}$$

References I

- Bellissima, Fabio (1986). “Finitely Generated Free Heyting Algebras”. In: *The Journal of Symbolic Logic* 51.1, pp. 152–165. ISSN: 00224812. JSTOR: 2273952.
- Grigolia, Revaz (1987). *Free algebras of non-classical logics.* Metsniereba Press.
- Łukasiewicz, Jan (1951). *Artistotle’s Syllogistic from the standpoint of modern formal logic.* Oxford: Clarendon Press.
- – (1952). “On the intuitionistic theory of deduction”. In: *Indagationes Mathematicae* 14, pp. 202–212.
- Rybakov, Vladimir V. (1994). “Criteria for admissibility of inference rules. Modal and intermediate logics with the branching property”. In: *Studia Logica* 53 (2), pp. 203–225. ISSN: 0039-3215. DOI: 10.1007/BF01054709.
- Shehtman, V.B. (1978). “Rieger-Nishimura Lattices”. In: *Soviet Mathematics Doklady* 19.