

A Syntactic Characterization of the Gabbay–de Jongh Logics

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$\neg A$ means that A is refutable



$\neg A$ means that A is refutable
there is a counter model to A



Łukasiewicz (1951)

$$\frac{}{\neg p} \quad \frac{\neg \sigma A}{\neg A} \quad \frac{A \vdash B \quad \neg B}{\neg A}$$



$\neg A$



$$\frac{-\sigma A}{-A}$$



$$\frac{\neg x \quad \sigma A \vdash x}{\neg \sigma A}$$

$$\neg A$$



Łukasiewicz (1952)

$$\frac{}{\neg p} \quad \frac{\neg \sigma A}{\neg A} \quad \frac{A \vdash B \quad \neg B}{\neg A}$$

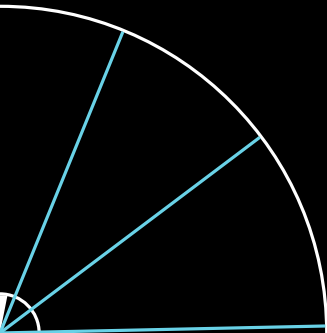
$$\frac{\neg A \quad \neg B}{\neg A \vee B}$$



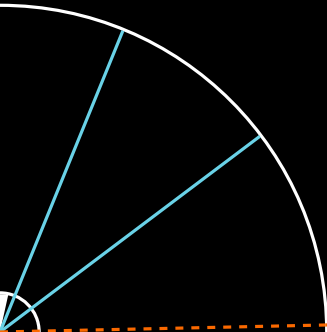
Overview



Overview



Overview



Syntactic Characterisation of BB_n

Overview

Admissible Reducibility

A diagram in the bottom-left corner of the slide. It features a white arc centered at the origin of a coordinate system. A solid blue line segment extends from the origin into the first quadrant. Two orange dashed lines also originate from the origin: one extends into the first quadrant, parallel to the blue line, and the other extends into the second quadrant. The text 'Admissible Reducibility' is written in orange, following the path of the dashed line in the second quadrant.

Syntactic Characterisation of BB_n

Overview

Admissible Reducibility

Expressing Extensions

Syntactic Characterisation of BB_n

A / Δ admissible



σA is derivable



A / Δ admissible



σC is derivable for some $C \in \Delta$



σA is derivable



$A \vdash \Delta$ admissible



σC is derivable for some $C \in \Delta$



$$\neg C \rightarrow A \vee B$$

$$(\neg C \rightarrow A) \vee (\neg C \rightarrow B)$$

$$\neg C \rightarrow A \vee B$$

$$\{ \neg C \rightarrow A, \quad \neg C \rightarrow B \}$$

\vdash is **admissibly reducible** to Θ

whenever

$\nVdash A$ iff $\sigma A \vdash \Theta$ for some σ



if \vdash is admissibly reducible to Θ
 then $\nabla = \vdash$.

$$\frac{D \in \Theta}{\vdash D} \quad \frac{\vdash \sigma A}{\vdash A} \quad \frac{\vdash C \text{ per } C \in \Delta \quad A \sim \Delta}{\vdash A}$$



Gabbay–de Jongh Logics

Logic of Bounded Branching

$$\text{BB}_n = \text{IPC} + \bigwedge_{i=0}^n \left(\left(x_i \rightarrow \bigvee_{j \neq i} x_j \right) \rightarrow \bigvee_{j \neq i} x_j \right) \rightarrow \bigvee_{i=0}^n x_i$$



$BB_n \not\models A$ iff there is a
proper & at most n -fold branching
tree T with $T \not\models A$.



Visser Rules

$$\frac{(\bigvee \Delta \rightarrow A) \rightarrow \bigvee \Delta}{\bigvee \{(\bigvee \Delta \rightarrow A) \rightarrow C \mid C \in \Delta\}}$$



Visser Rules

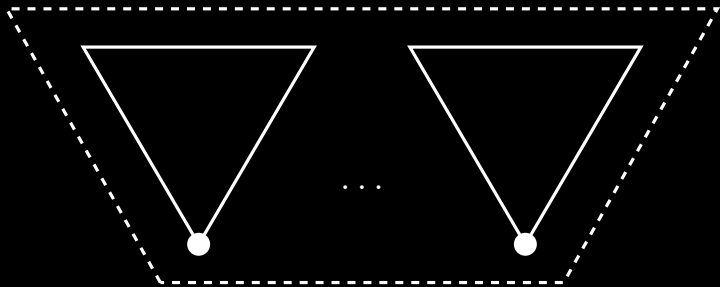
$$(\bigvee \Delta \rightarrow A) \rightarrow \bigvee \Delta$$

$$\{(\bigvee \Delta \rightarrow A) \rightarrow C \mid C \in \Delta\}$$

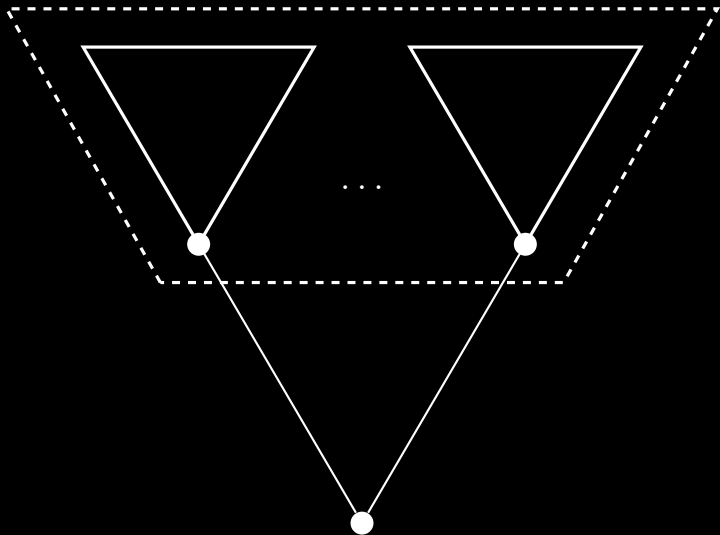
Extension Property



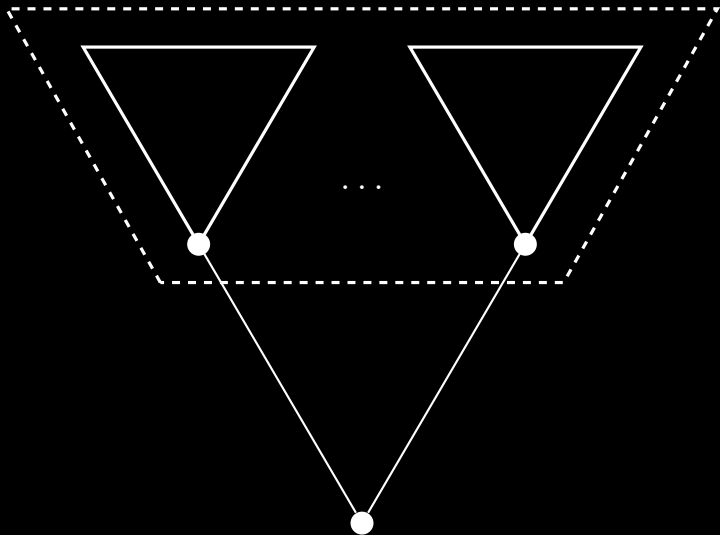
Extension Property



Extension Property



n^{th} Extension Property



Jankov–de Jongh formulae

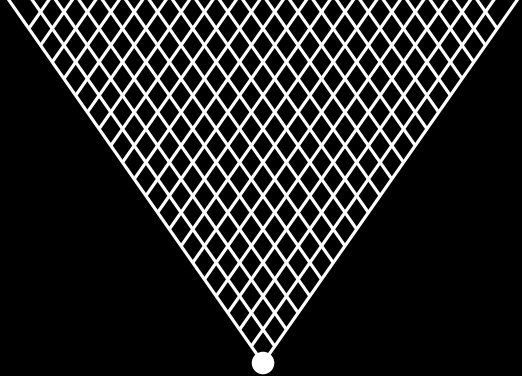
In suitable models have

$$l \Vdash \text{up } k \quad \text{iff} \quad k \leq l$$

$$l \Vdash \text{nd } k \quad \text{iff} \quad l \not\leq k$$

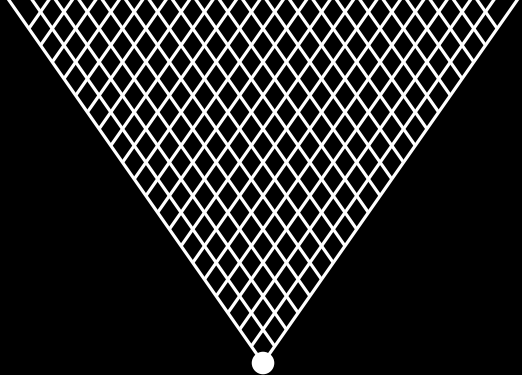


•
k



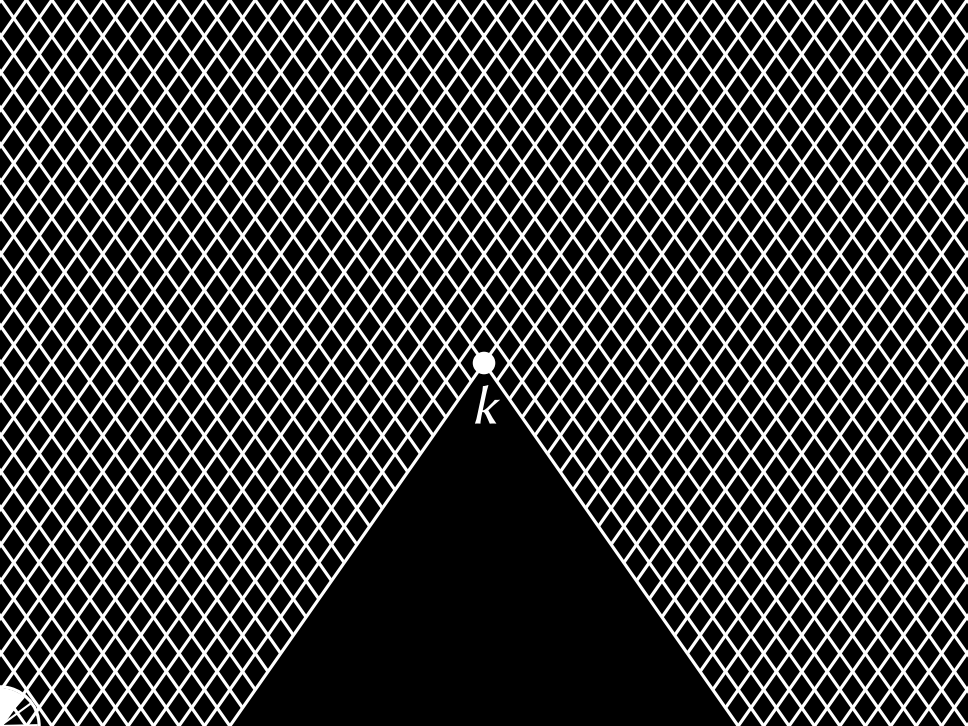
k



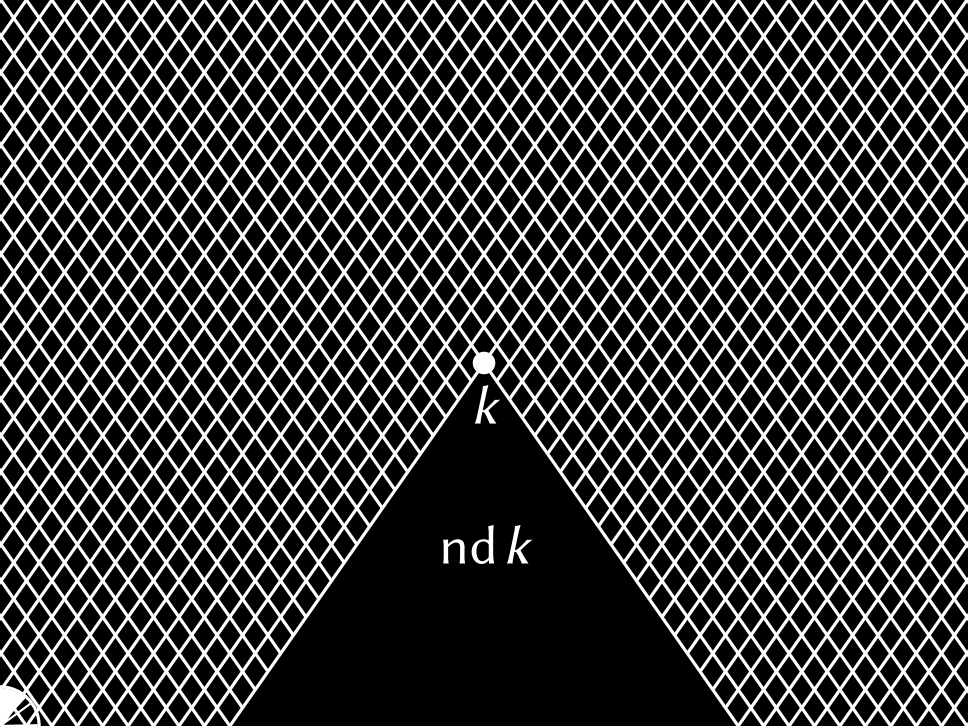


k

up k



k

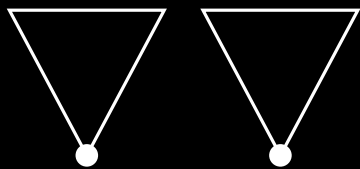


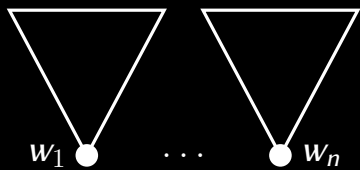
Per X there is a suitable model
which “contains a copy” of each
finite model on X .

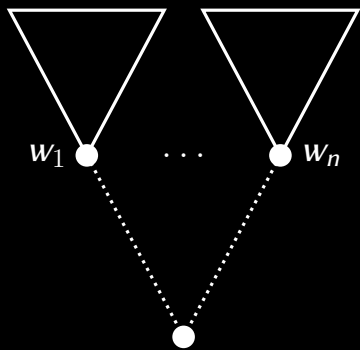


Per X there is a suitable model
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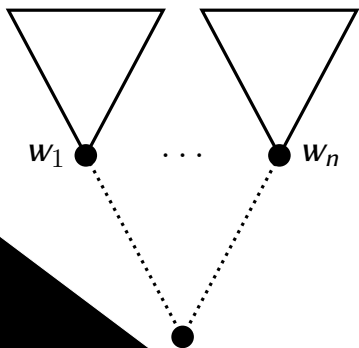
Bellissima (1986) · Rybakov (1994)
Shehtman (1978) · Grigolia (1987)



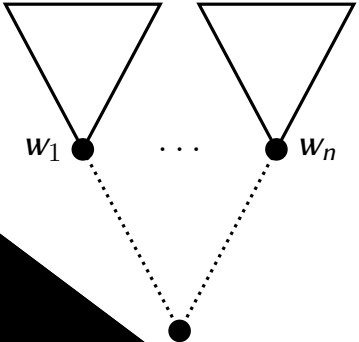




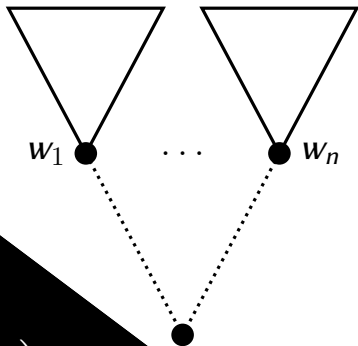
semantics



semantics
syntax



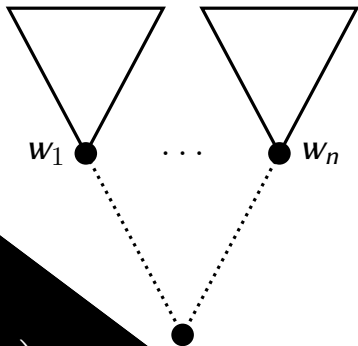
semantics
syntax



$$\vdash \bigvee_{j=1}^n \left(\bigvee_{i=1}^n \text{nd } w_i \rightarrow \bigvee_{i=1}^n \text{up } w_i \right) \rightarrow \text{nd } w_j$$

$$\text{or } \not\vdash \left(\bigvee_{i=1}^n \text{nd } w_i \rightarrow \bigvee_{i=1}^n \text{up } w_i \right) \rightarrow \bigvee_{i=1}^n \text{nd } w_i$$

semantics
syntax

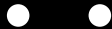


$$\vdash \bigvee_{C \in \Delta} \left(\bigvee \Delta \rightarrow A \right) \rightarrow C$$

$$\text{or } \not\vdash \left(\bigvee \Delta \rightarrow A \right) \rightarrow \bigvee \Delta$$

A logic with DP and FMP
admits the Visser rules up to n
iff it has
the extension property up to n .

Analogous



Analogous

The image features a black background with the word "Analogous" centered at the top in a white serif font. Below the text, two small white circular dots are positioned horizontally. From each dot, two white lines extend upwards and outwards, crossing each other in the center to form two overlapping V-shapes. The lines from the left dot extend towards the top-left and top-right, while the lines from the right dot extend towards the top-right and top-left.

Analogous



The diagram consists of two V-shaped structures, one on the left and one on the right, both pointing downwards. They meet at a central point. Below this meeting point, there are two small white dots. The word "Analogous" is centered at the top of the image.

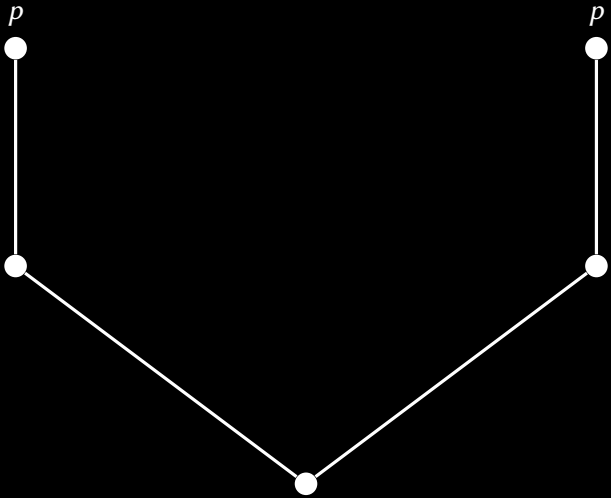
Analogous

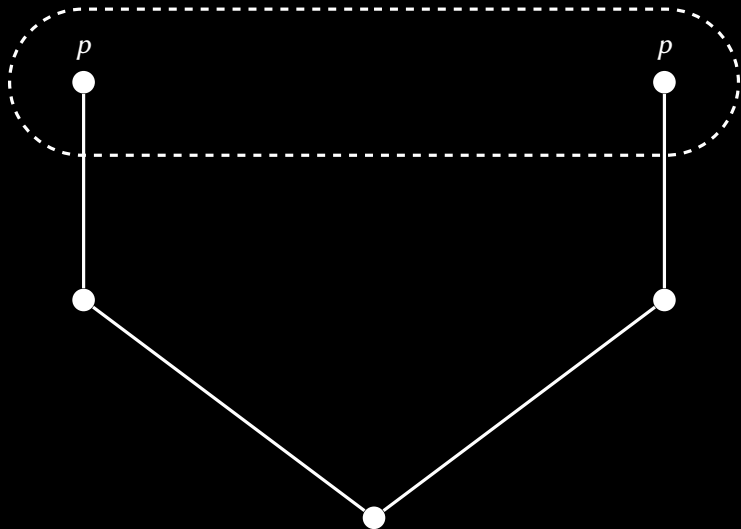
A diagram consisting of a large inverted V-shape formed by two white lines extending from the top corners towards a central point. At the bottom of this V-shape, there is a small, upward-curving arc. Below this arc, there are two small white dots positioned horizontally next to each other.

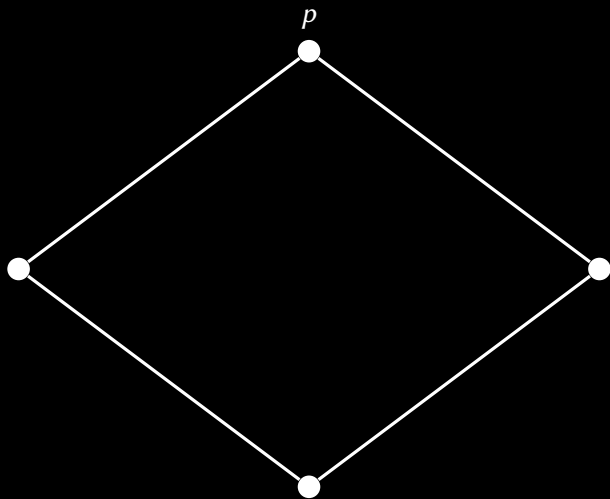
Analogous

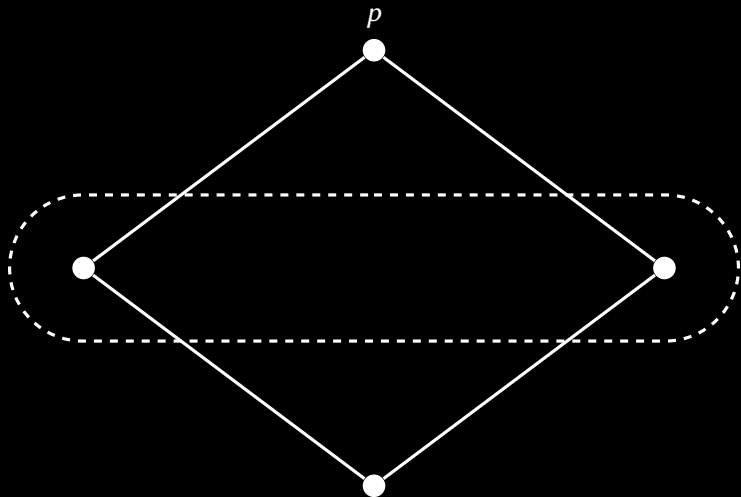


$k \equiv l$ when $v(k) = v(l)$ and $k \leq u$ iff $l \leq u$ for all $u \neq k, l$



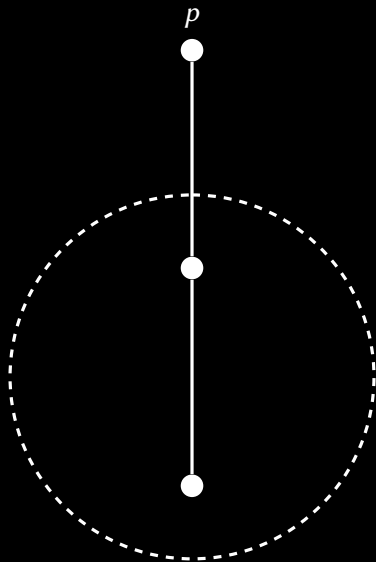






p





p



$$t_T : T \rightarrow PT$$

$t_T, k \Vdash l$ iff $l = k$ and k is maximal

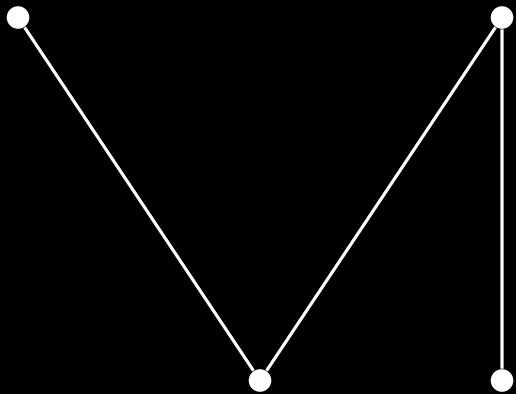
A poset T is called
maximally distinguishable
if in the model t_T
analogous is equal.

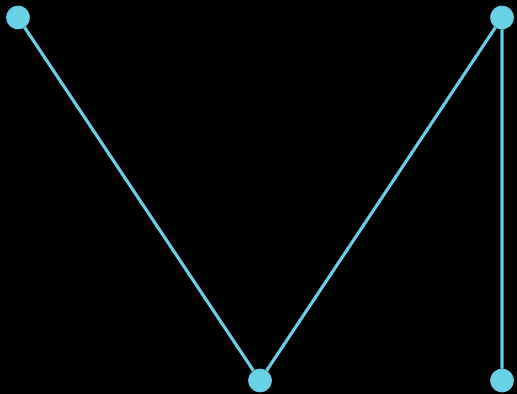
$$t_T : T \rightarrow PT$$

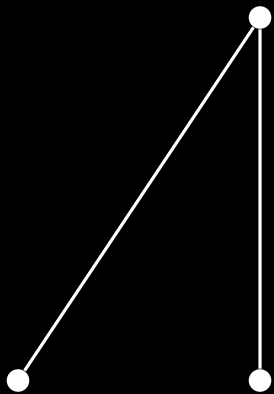
$t_T, k \Vdash l$ iff $l = k$ and k is maximal

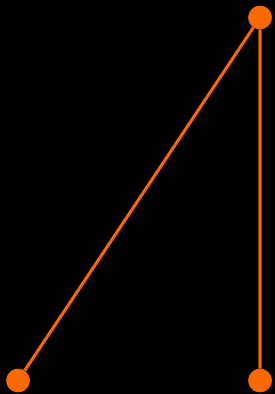












Every proper tree is
maximally distinguishable.

If T is proper & maximally
distinguishable then
 $\text{nd } t_T \sim \{\text{nd } w \mid w \in T \text{ maximal}\}$.

$$BB_n \not\equiv A$$

$$\text{BB}_n \not\vdash A$$

$$T \not\vdash A$$

$BB_n \not\vdash A$

$T \not\vdash A$

$\sigma A \vdash \text{nd } t_T$

$$\text{BB}_n \not\vdash A$$
$$T \not\vdash A$$
$$\sigma A \vdash \text{nd } t_T$$
$$\sim \{ \text{nd } w \mid w \in T \text{ maximal} \}$$







BB_n reduces admissibly to
CPC-non-derivable formulae

$$\frac{}{\neg p} \quad \frac{\neg \sigma A}{\neg A} \quad \frac{A \vdash B \quad \neg B}{\neg A}$$

$$\frac{\neg (\bigvee_{i=1}^n C_i \rightarrow A) \rightarrow C_j \text{ for all } j}{\neg (\bigvee_{i=1}^n C_i \rightarrow A) \rightarrow \bigvee_{j=1}^n C_j}$$



References I

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-  **Shehtman, V.B. (1978)**. “Rieger-Nishimura Lattices”. In: *Soviet Mathematics Doklady* 19.