Characterizing Admissible Rules

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June 13th 2012

$A \vee B$ derivable

$$\frac{A \vee B \text{ derivable}}{A \text{ or } B \text{ derivable}}$$

$$\frac{\neg C \to A \lor B}{(\neg C \to A) \lor (\neg C \to B)}$$

Γ / Δ admissible

A^σ is derivable for each $A \in \Gamma$

$\Gamma \,/\, \Delta$ admissible

B^σ is derivable for some $B \in \Delta$

Disjunction Again

$$\frac{\{\ \Rightarrow p \vee q\ \}}{\{\ \Rightarrow p,\ \Rightarrow q\ \}}$$

Disjunction Again

$$\frac{\Rightarrow p \vee q}{\Rightarrow p, \ \Rightarrow q}$$

Disjunction Again

$$\frac{\neg a \Rightarrow p \vee q}{\neg a \Rightarrow p, \neg a \Rightarrow q}$$

Disjunction Again

$$\frac{a \to \bot \Rightarrow p \vee q}{a \to \bot \Rightarrow p, a \to \bot \Rightarrow q}$$

Visser Rules

$$\frac{\Rightarrow p \vee q}{\{\Rightarrow p, \Rightarrow q\}}$$

Visser Rules

$$\frac{\Rightarrow \Delta}{\{ \quad \Rightarrow \Pi \}_{\Pi \in \Delta}}$$

Visser Rules

$$\dfrac{\begin{array}{c}\{\ a_i \to b_i\ \}_{i=1}^{n} \\ \vert \\ \Gamma \Rightarrow \Delta\end{array}}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta} \cup \{\ \Gamma \Rightarrow a_i\ \}_{i=1}^{n}}$$

Visser Rules

$$\{\ a_i \to b_i\ \}_{i=1}^{n}$$

$$\Big|$$

$$\Gamma \Rightarrow \Delta$$

$$\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta} \cup \{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Gamma^{\mathrm{a}}}$$

Visser Rules

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta \cup \Gamma^a}}$$

Visser Rules

$$\frac{\left(\bigwedge_{i=1}^{n} A_i \to B_i\right) \to A_{n+1} \vee A_{n+2}}{\bigvee_{j=1}^{n+2} \left(\bigwedge_{i=1}^{n} A_i \to B_i\right) \to A_j}$$

Friedman (1975)

Can one decide which single-conclusion rules are admissible rules in IPC?

Rybakov (1984)

Yes

Visser & de Jongh

Do the Visser rules characterize admissibility?

Iemhoff (2001) & Rozière (1992)

Yes

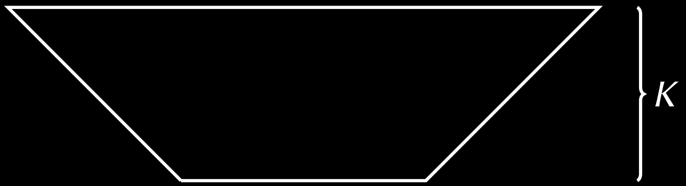
What about intermediate logics?

Iemhoff (2005)

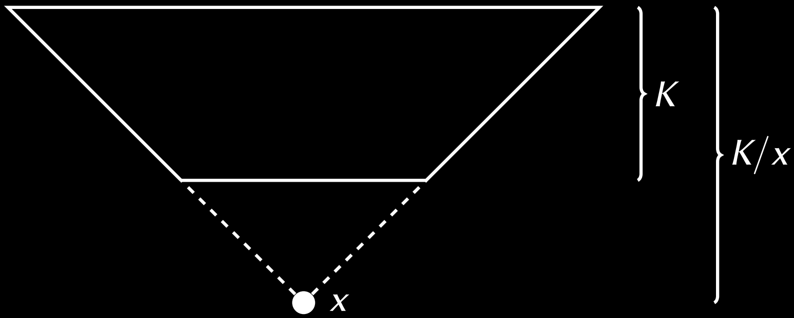
When the Visser rules are admissible, they characterize.

And when they are not?

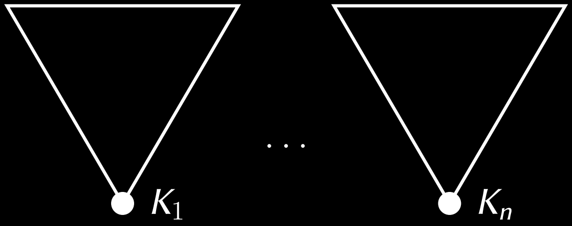
Extension



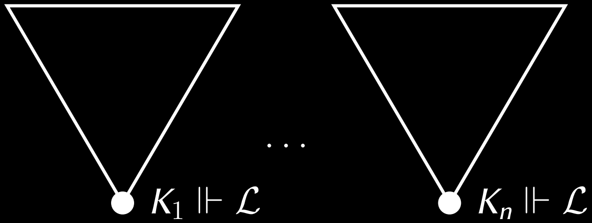
Extension



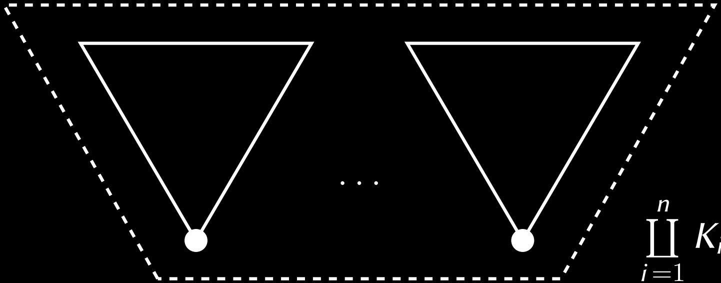
Extension property



Extension property

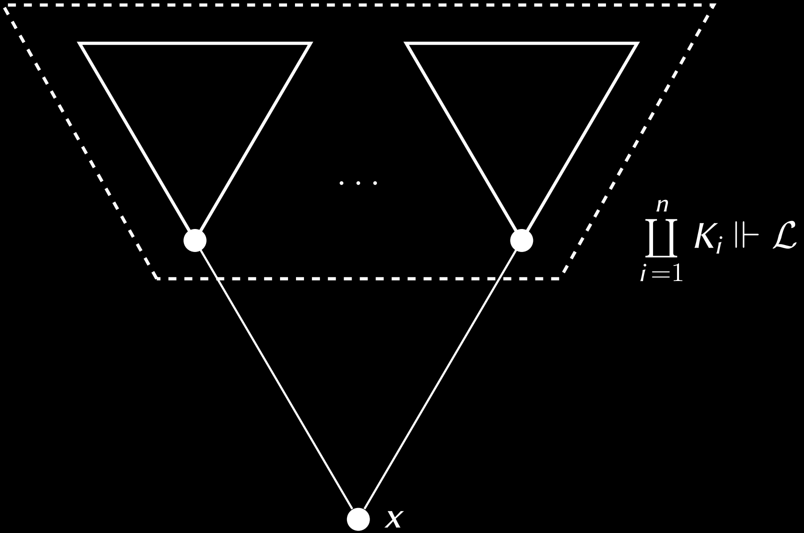


Extension property

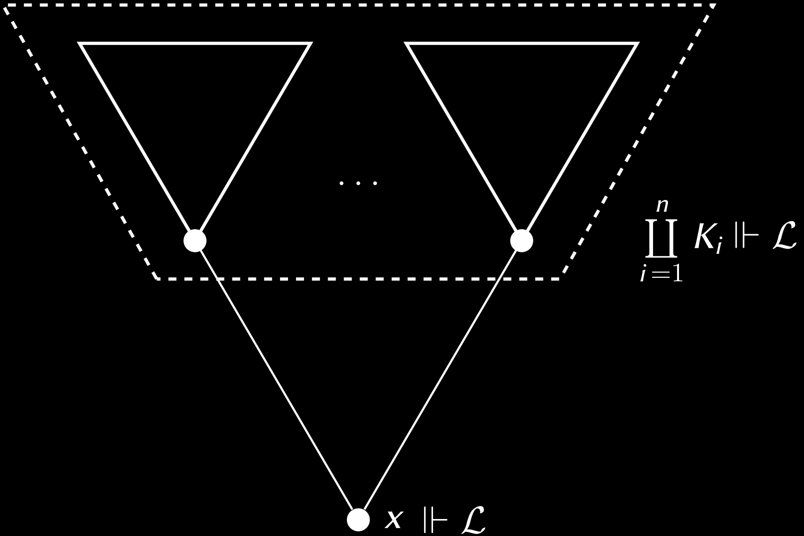


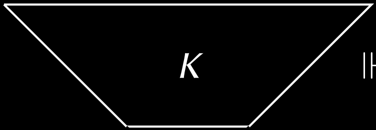
$$\coprod_{i=1}^{n} K_i \Vdash \mathcal{L}$$

Extension property

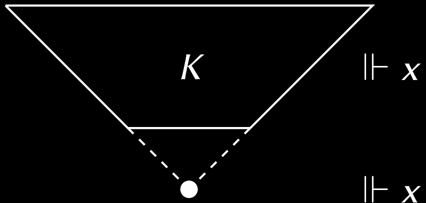


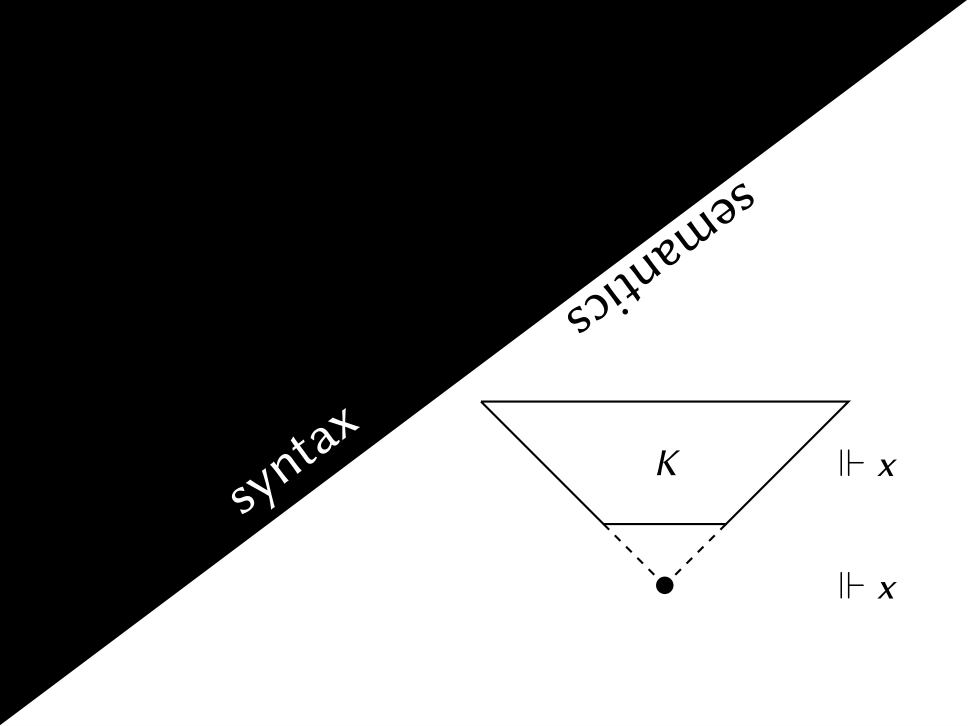
Extension property





$\Vdash x$


K
⊩ x
⊩ x

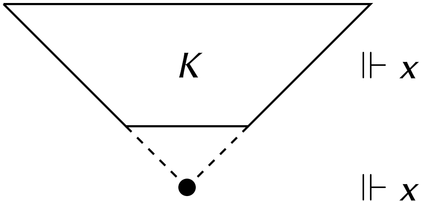

syntax
semantics
K
$\Vdash x$
$\Vdash x$

If $x + \left\{ A \to B \;\middle|\; \begin{array}{l} K \Vdash A \to B \\ K \nVdash A \end{array} \right\} \vdash \bigvee \Delta$

then Δ hits $\mathbf{Th}\, K$

semantics

syntax



de Jongh Rules

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \,\in\, \Delta \,\cup\, \mathcal{U}}}$$

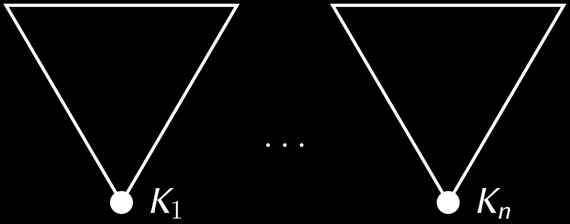
de Jongh Rules

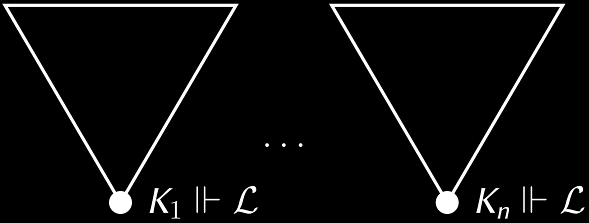
$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi\ \in\ \Delta\ \cup\ \mathcal{U}}}$$

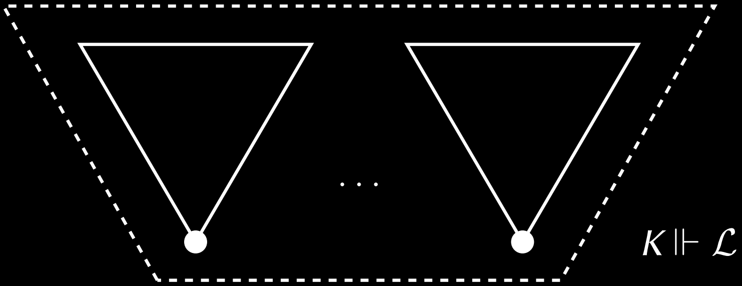
covers Γ^{a} & has *n* elements

Gabbay and de Jongh (1974): Logic of at most $(n+1)$-fold branching finite trees

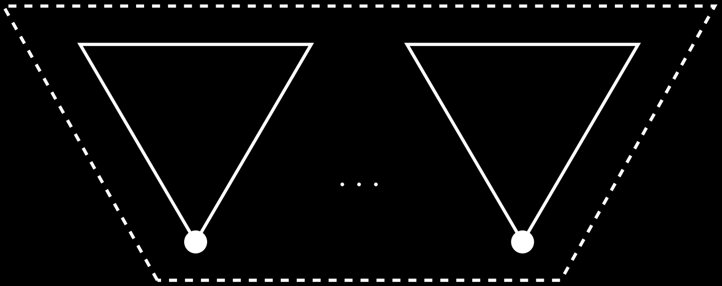
$(n + 1)^{\text{th}}$ de Jongh rule admissible in n^{th} Gabby–de Jongh Logic


K_1
...
K_n

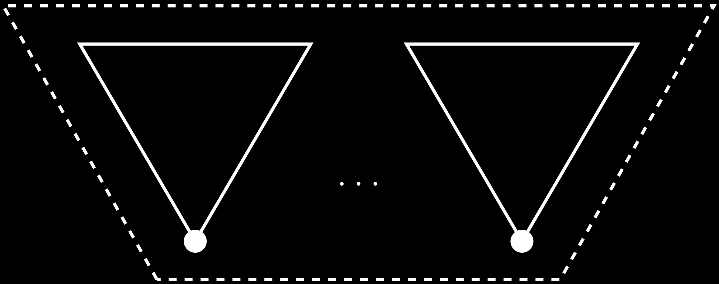

$K_1 \Vdash \mathcal{L}$
$\cdots$
$K_n \Vdash \mathcal{L}$



$K \Vdash \mathcal{L}$

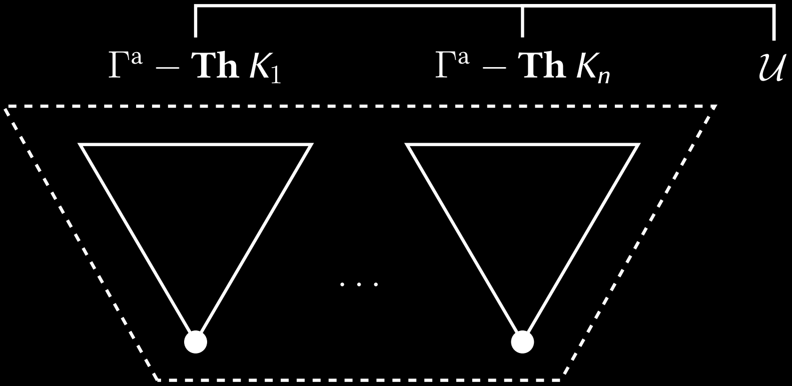


$$\left\{ A \to B \;\middle|\; \begin{matrix} K \Vdash A \to B \\ K \nVdash A \end{matrix} \right\} \quad \vdash_{\mathcal{L}} \bigvee \Delta$$



$$\left\{ A \to B \;\middle|\; \begin{array}{l} K \Vdash A \to B \\ K \nVdash A \end{array} \right\} \supseteq \Gamma \vdash_{\mathcal{L}} \bigvee \Delta$$

finite



$$\left\{ A \to B \;\middle|\; \begin{array}{l} K \Vdash A \to B \\ K \nVdash A \end{array} \right\} \supseteq \Gamma \vdash_{\mathcal{L}} \bigvee \Delta$$

finite

Intermediate logic with DP:

n^{th} de Jongh rule admissible

iff

have n^{th} extension property

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