Universal Model

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$\dashv A$ means that A is refutable

Refutation System

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

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Łukasiewicz (1951)

$A \in \mathcal{L}(X)$ classically derivable iff
$v \models A$ for all $v : X \to \{0, 1\}$.

Intuitionistic Propositional Logic

Intuitionistic Propositional Logic

IPC

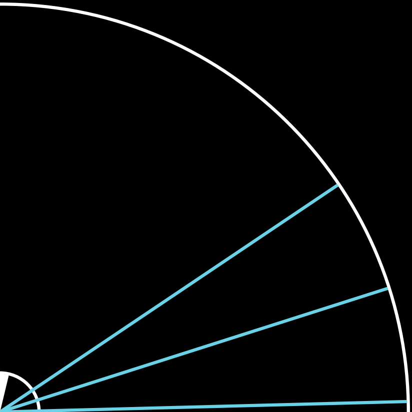
Łukasiewicz (1952)

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

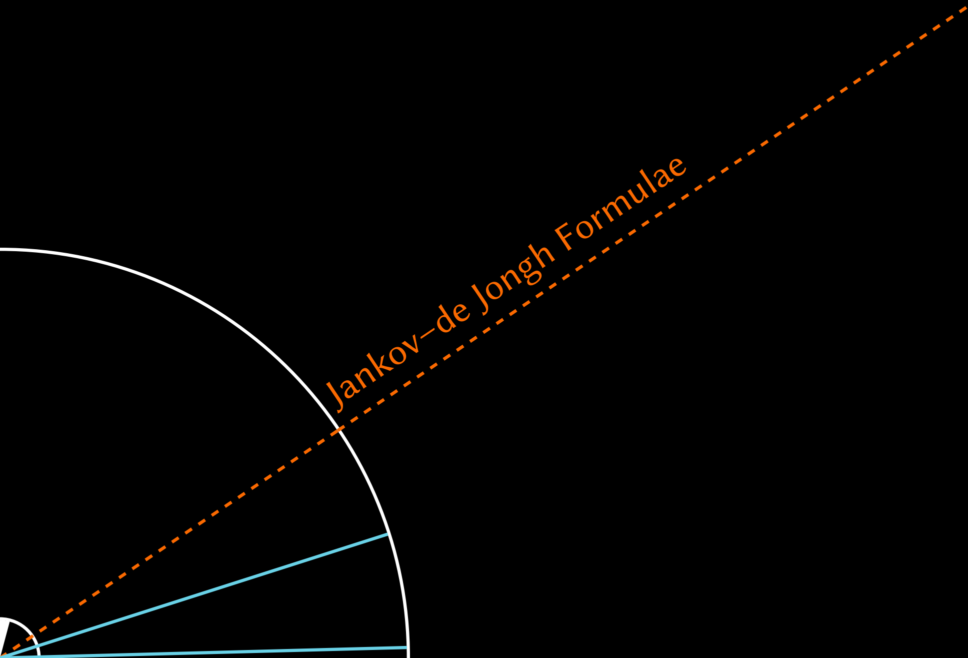
$$\frac{\dashv A \qquad \dashv B}{\dashv A \vee B}$$

Overview

Overview



Overview



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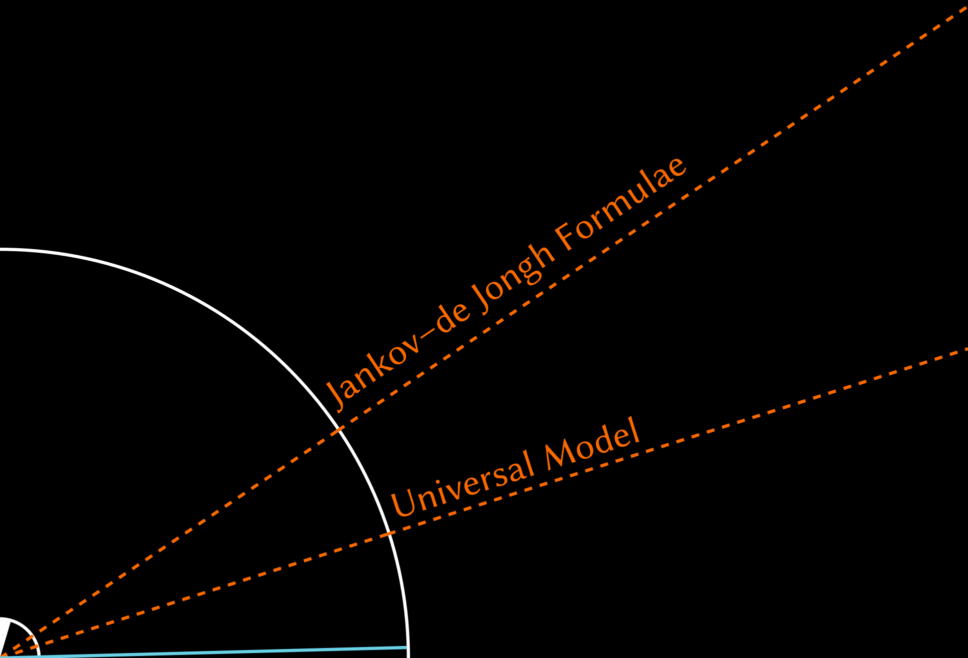

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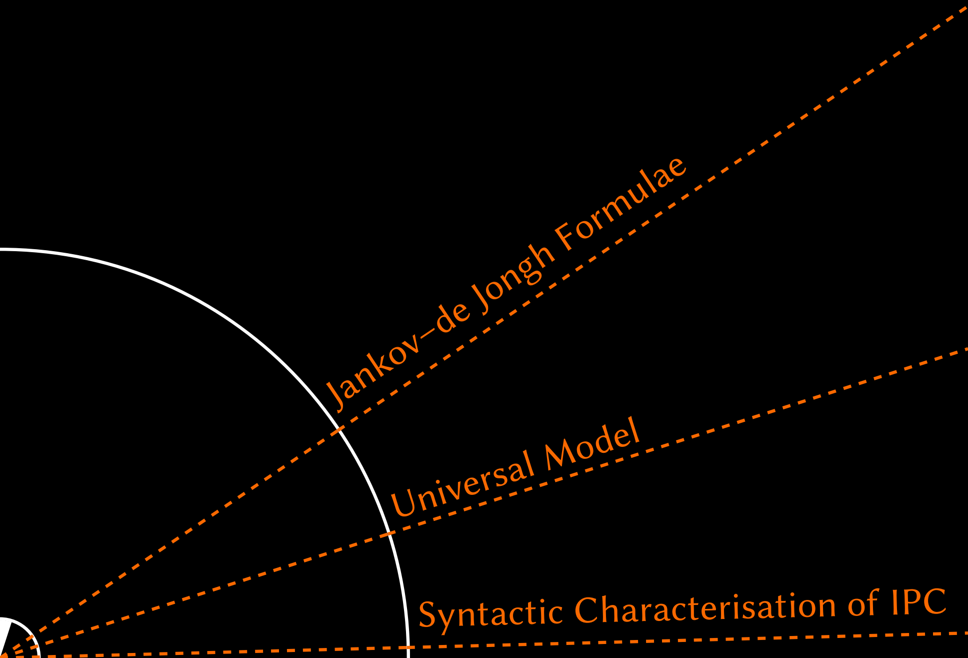


Jankov–de Jongh Formulae
Universal Model
Syntactic Characterisation of IPC

Kripke models

Kripke models

reflexive

Kripke models

reflexive · transitive

Kripke models

reflexive · transitive · anti-symmetric

Kripke models

reflexive · transitive · anti-symmetric
preservation of truth

Image-finite

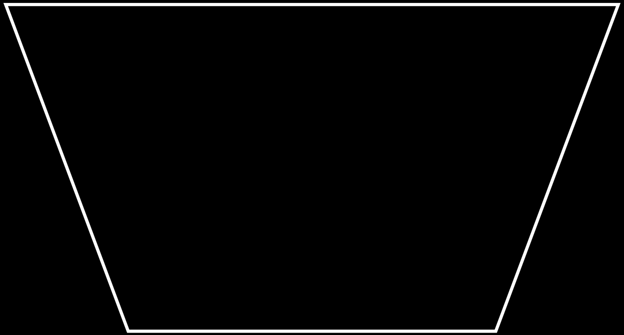


Image-finite

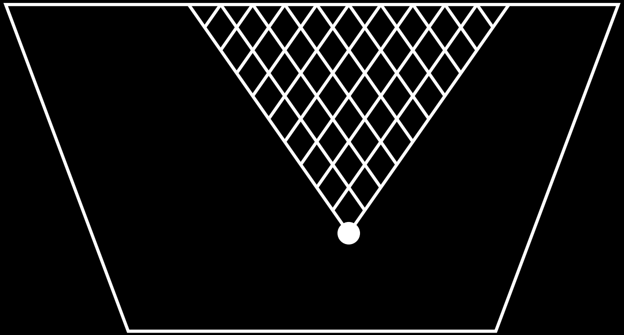
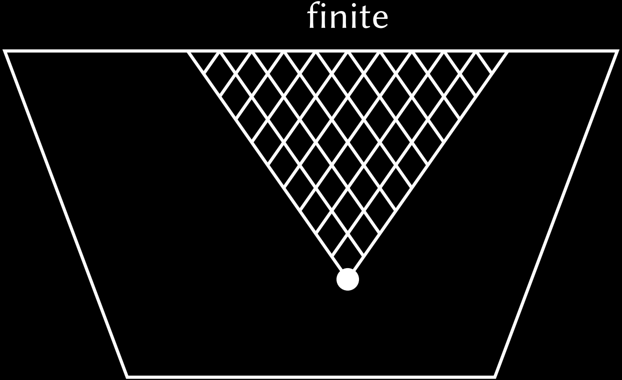
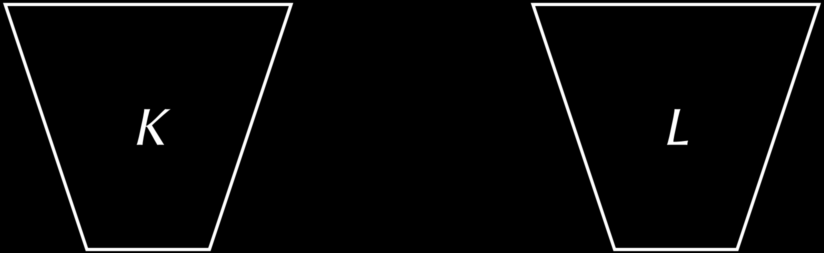


Image-finite

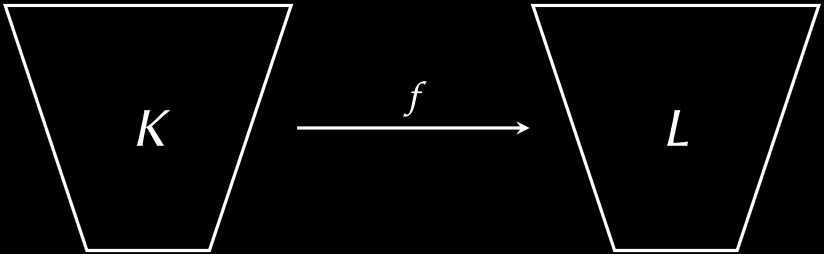


Morphisms

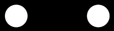
Morphisms



Morphisms



Analogous



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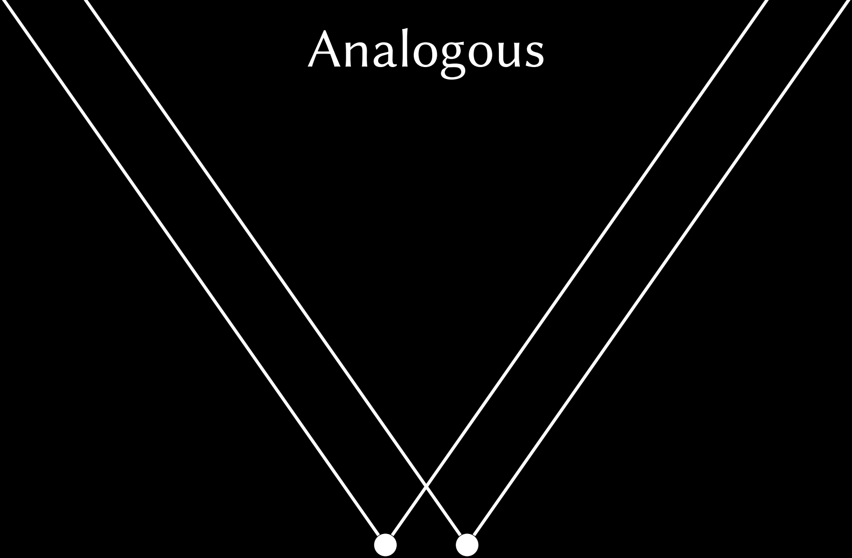

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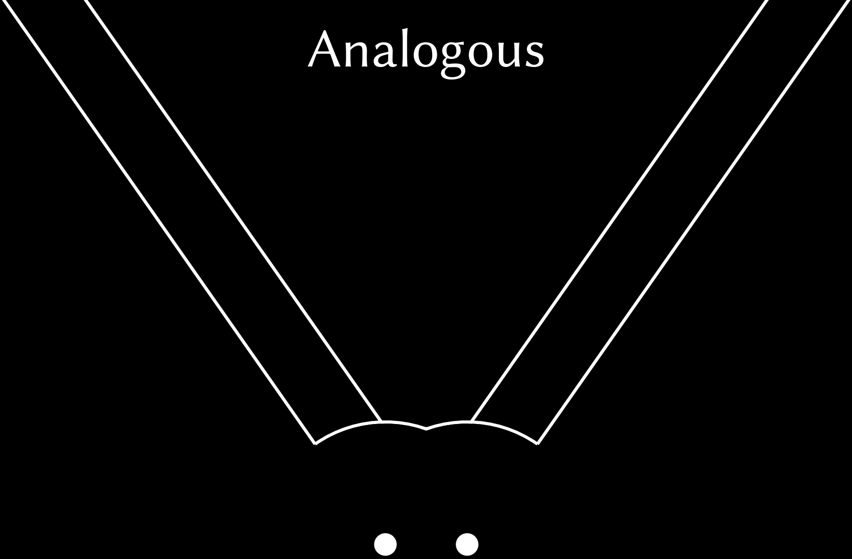

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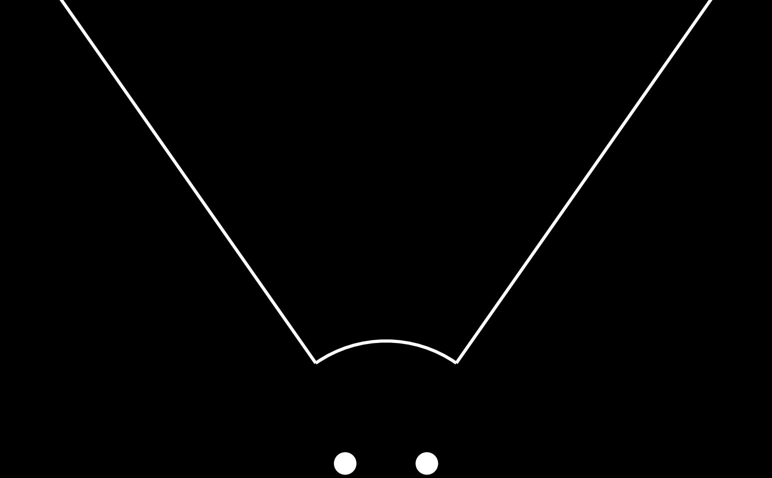

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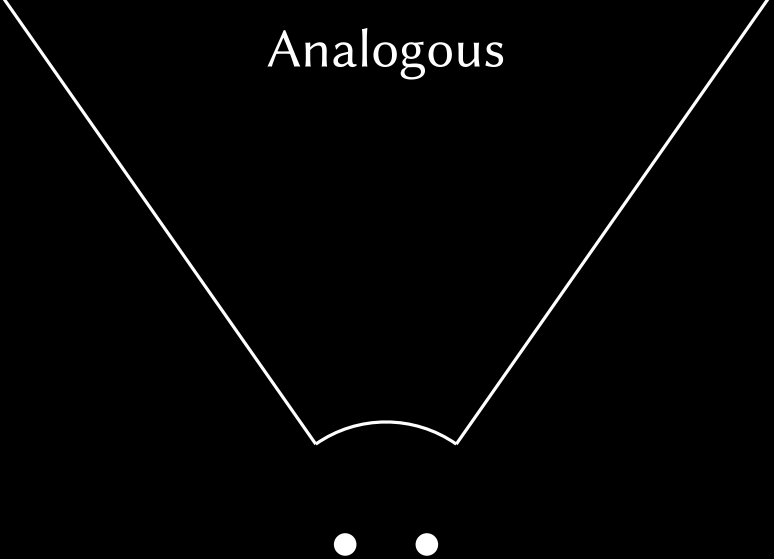
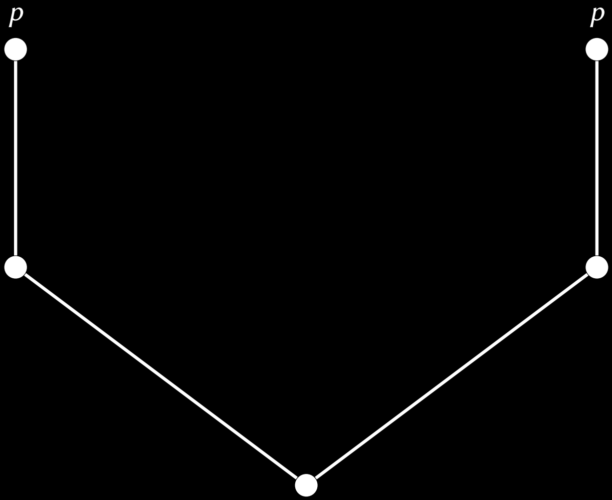
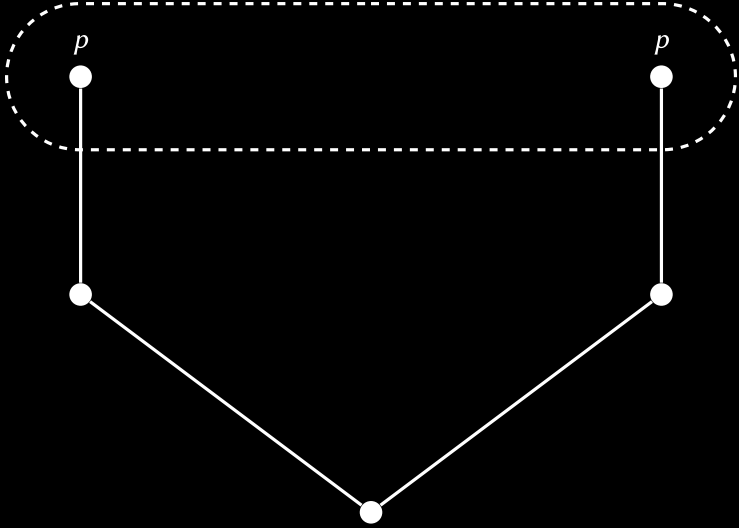


$k \equiv l$ when $v(k) = v(l)$ and $k \leq u$ iff $l \leq u$ for all $u \neq k, l$


p
p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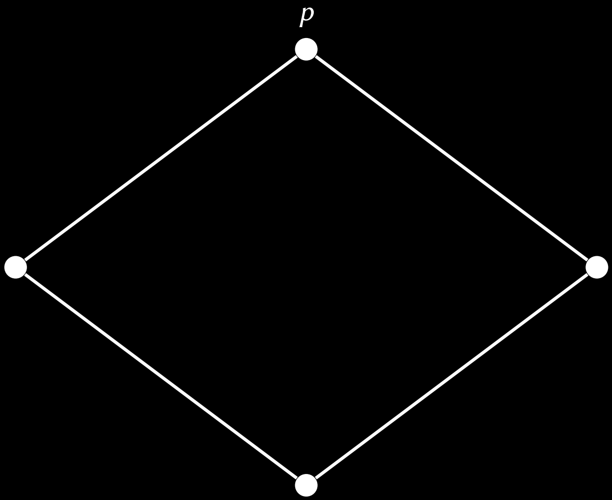
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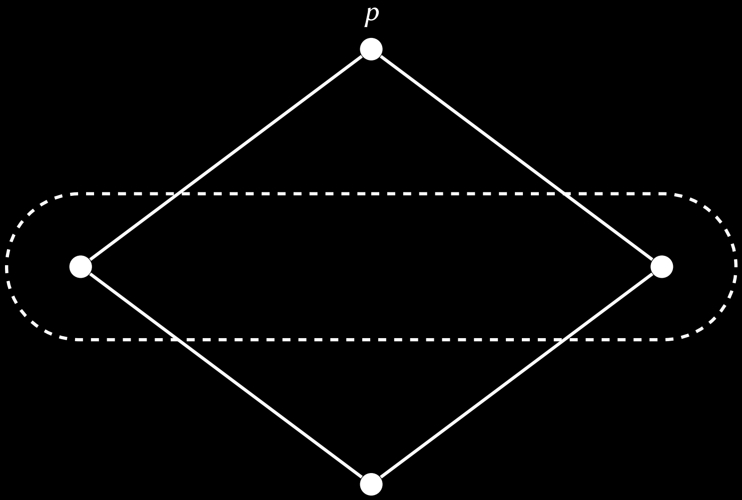
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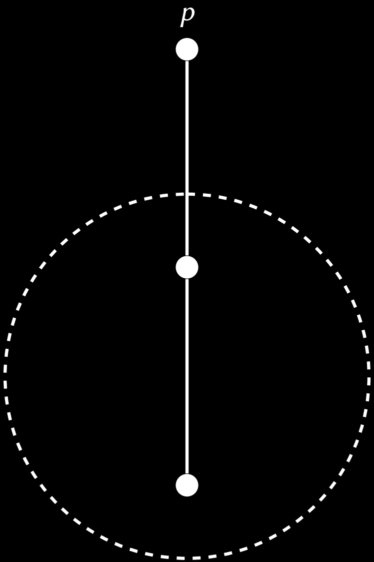
p


p


p

A model is concrete when analogous nodes are equal.

In concrete & image-finite models, upsets & complements of downsets are definable.

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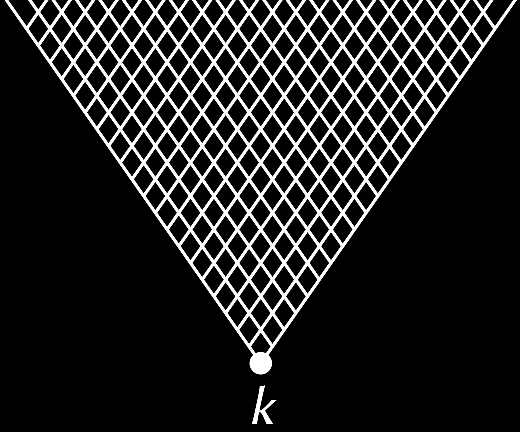
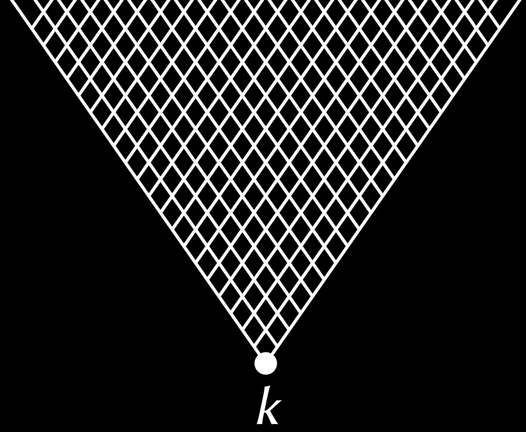
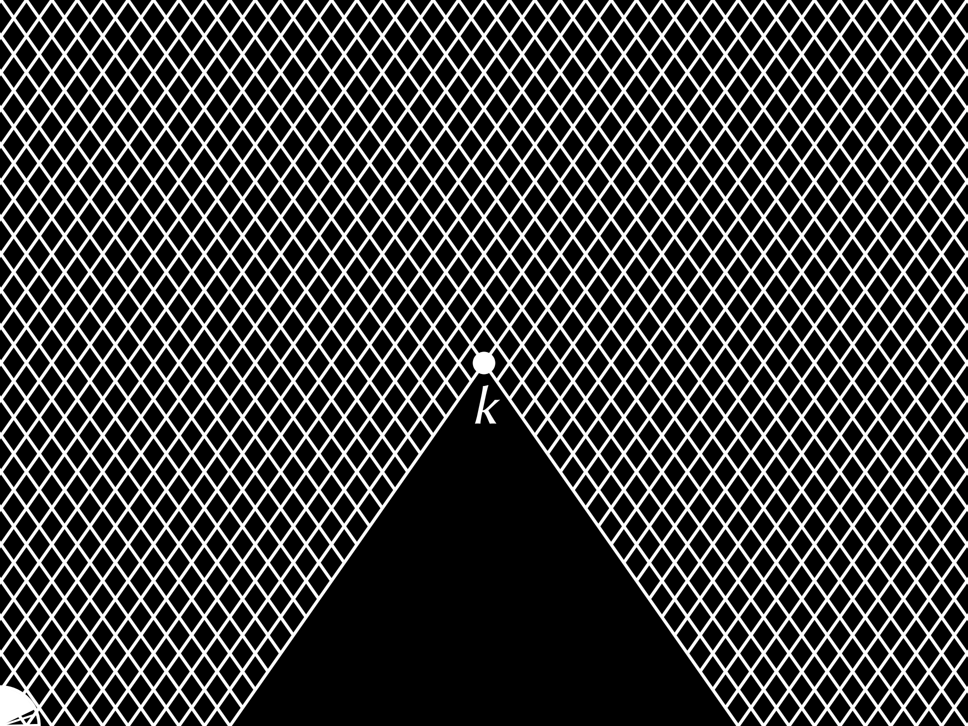
de Jongh (1968)

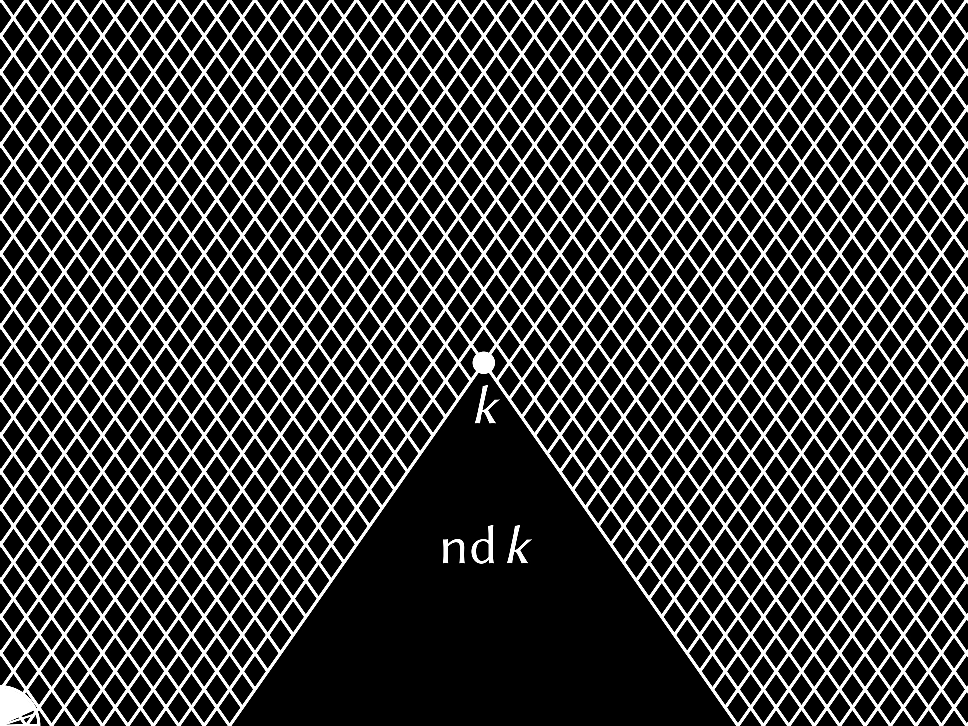
k


k



up k


k


k
nd k

$$\operatorname{up} k := \bigwedge_{k \Vdash p} p \wedge \left(\bigvee_{\substack{W \Vdash p \\ k \nVdash p}} p \vee \bigvee_{w \in W} \operatorname{nd} w \rightarrow \bigvee_{w \in W} \operatorname{up} w \right)$$

$$\mathrm{up}\, k := \bigwedge_{k \Vdash p} p \wedge \left(\bigvee_{\substack{W \Vdash p \\ k \nVdash p}} p \vee \bigvee_{w \in W} \mathrm{nd}\, w \to \bigvee_{w \in W} \mathrm{up}\, w \right)$$

$$\mathrm{nd}\, k := \mathrm{up}\, k \to \bigvee_{w \in W} \mathrm{up}\, w$$

A model is universal
when it is
the terminal image-finite model.

A model is universal over X when it is the terminal image-finite model over X.

The universal model is complete.

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Bellissima (1986) · Rybakov (1994)

The universal model is complete.

Bellissima (1986) · Rybakov (1994)
Shehtman (1978) · Grigolia (1987)

$$\mathrm{Univ}\, X \cong \mathrm{colim} \begin{pmatrix} \mathrm{FRKripke}\, X \\ \text{\rotatebox{90}{$\subseteq$}} \\ \mathrm{Kripke}\, X \end{pmatrix}$$

The universal model is concrete.

The image-finite part of the canonical model is the universal model.

$$A \to \mathrm{nd}\, k$$

$$A \to \mathrm{nd}\, k$$

for all $l \Vdash A$ have $l \not\leq k$

$$A \to \mathrm{nd}\, k$$

for all $l \Vdash A$ have $l \not\leq k$

$$k \nVdash A$$

Skura (1989)

$$\frac{}{\dashv p} \qquad \frac{\dashv \sigma A}{\dashv A} \qquad \frac{A \vdash B \qquad \dashv B}{\dashv A}$$

$$\frac{\dashv (\bigvee B_i \to A) \to B_j \text{ for all } j}{\dashv (\bigvee B_i \to A) \to \bigvee B_i}$$

IPC is the intermediate logic where

$$\nvdash \; = \; \dashv$$

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