Admissible Rules

Saturation: Syntax & Semantics

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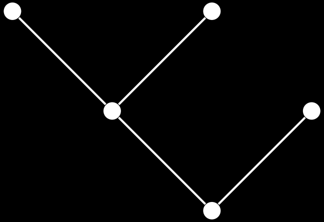
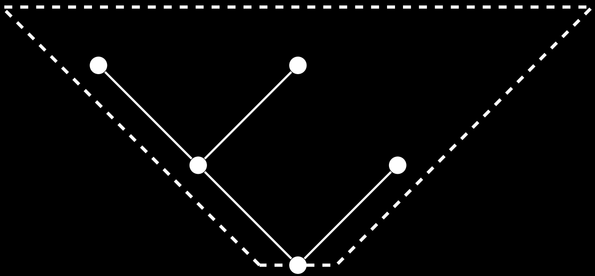
Intuitionistic Logic

Kripke Semantics



Kripke Semantics



Disjunction Property

$$\frac{\vdash \phi \vee \psi}{\vdash \phi \;\text{ or }\; \vdash \psi}$$

Disjunction Property Extended

$$\neg\chi \vdash \phi \vee \psi$$

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$$\frac{\neg\chi \vdash \phi \vee \psi}{\neg\chi \vdash \phi \ \text{ or } \ \neg\chi \vdash \psi}$$

Admissible Rule

Definition
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for all sequents $\Gamma \Rightarrow \Delta$ in S it holds that $\sigma(\Gamma) \vdash \bigvee \sigma(\Delta)$

Disjunction Again

$$\frac{\{\ \Rightarrow p \vee q\ \}}{\{\ \Rightarrow p,\ \Rightarrow q\ \}}$$

Disjunction Again

$$\frac{\Rightarrow p \vee q}{\Rightarrow p, \;\Rightarrow q}$$

Disjunction Again

$$\frac{\neg a \Rightarrow p \vee q}{\neg a \Rightarrow p, \neg a \Rightarrow q}$$

Disjunction Again

$$\frac{a \to \bot \Rightarrow p \lor q}{a \to \bot \Rightarrow p, a \to \bot \Rightarrow q}$$

Visser Rules

$$\frac{\Rightarrow p \vee q}{\{\Rightarrow p, \Rightarrow q\}}$$

Visser Rules

$$\frac{\Rightarrow \Delta}{\{ \quad \Rightarrow \Pi \}_{\Pi \in \Delta}}$$

Visser Rules

$$\{\ a_i \to b_i\ \}_{i=1}^{n}$$

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta} \cup \{\ \Gamma \Rightarrow a_i\ \}_{i=1}^{n}}$$

Visser Rules

$$\{\ a_i \rightarrow b_i\ \}_{i=1}^{n}$$

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta} \cup \{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Gamma^{\mathrm{a}}}}$$

Visser Rules

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta\, \cup\, \Gamma^{a}}}$$

Visser rules characterize admissibility

Classical

Intuitionistic

Classical

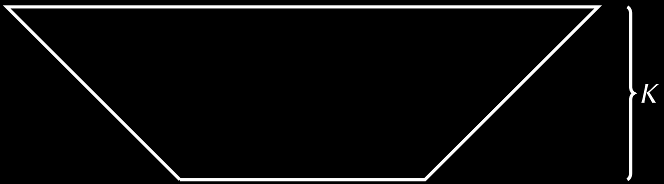
Intermediate

Intuitionistic

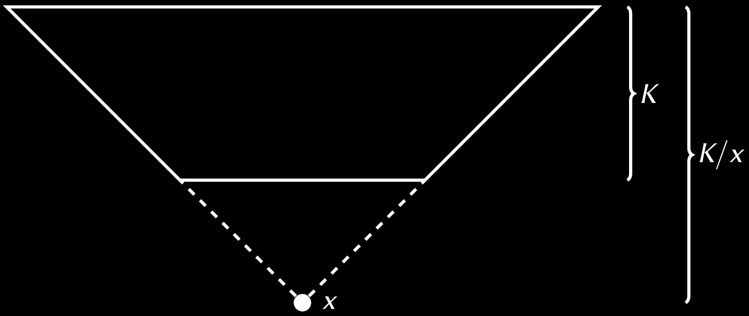
Iemhoff (2005):

An intermediate logic admits the Visser rules
if and only if
it has the weak extension property

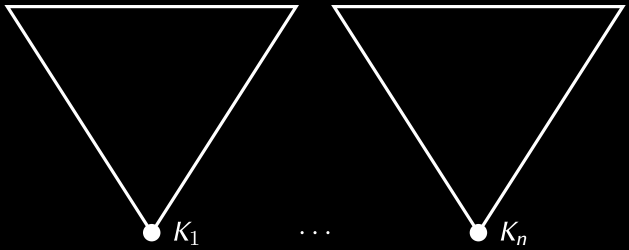
Extension



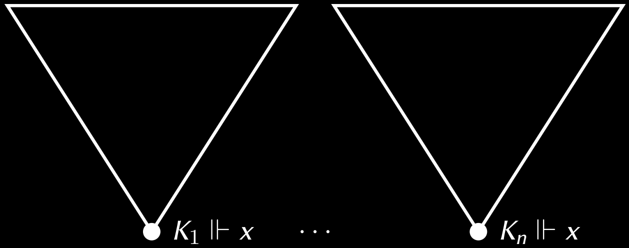
Extension



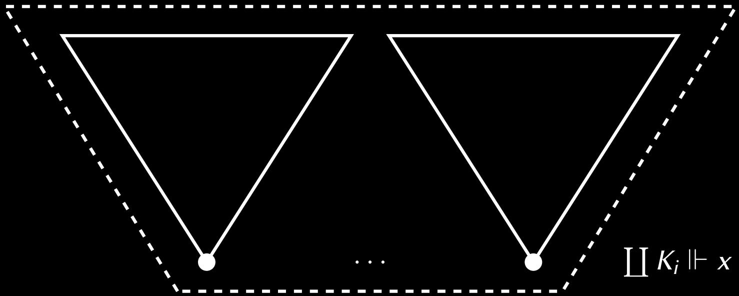
Extension property



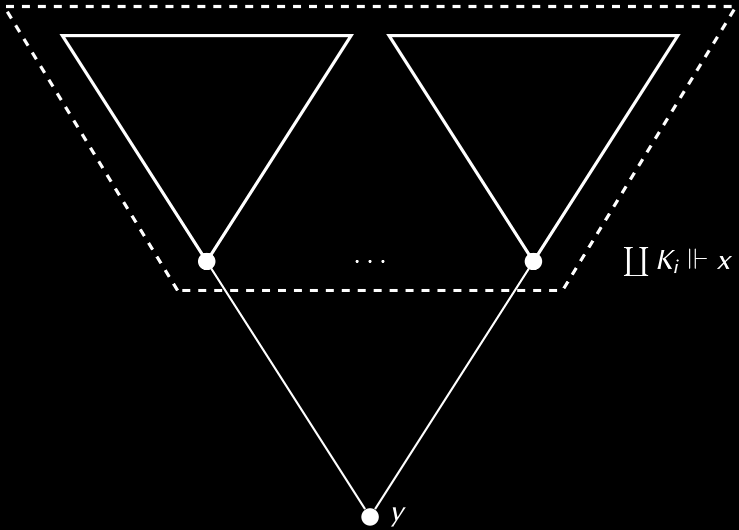
Extension property



Extension property



Extension property



Vacuous Implications

$$\mathrm{I}\,x := \left\{ \phi \to \psi \;\middle|\; \begin{array}{l} x \vdash \psi \\ x \nvdash \phi \end{array} \right\}$$

Saturation

x is saturated in y whenever
if $x \vdash \bigvee \Delta$ then Δ intersects y

Th $K/x + \mathrm{I}\,K$ is saturated in K

$$\underbrace{\mathbf{Th}\, K/x}_{\mathsf{x}} + \mathrm{I}\, K \text{ is saturated in } K$$

Characterization

Let K be a model and x be saturated $\mathbf{Th}\, K$. These are equivalent:

1. Some extension of K forces x;
2. The set $x \cup \mathrm{I}\, K$ is saturated in $\mathbf{Th}\, K$;
3. A tight predecessor of $\mathbf{Th}\, K$ above x exists.

de Jongh Rules

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi\ \in\ \Delta\ \cup\ \mathcal{U}}}$$

de Jongh Rules

$$\frac{\Gamma \Rightarrow \Delta}{\{\ \Gamma \Rightarrow \Pi\ \}_{\Pi \in \Delta \cup \mathcal{U}}}$$

$|\ \mathcal{U}\ | = n$ and $\bigcup \mathcal{U} = \Gamma^{\mathrm{a}}$

An intermediate logic with disjunction property

admits the n^{th} de Jongh rule

if and only if

it has the n^{th} extension property

If *n* models can be
extended with a root,
the de Jongh rules
hold to boot.