Characterising Logics through their Admissible Rules

Jeroen Goudsmit
Utrecht University
July 14^{th} 2014, 17:30 – 17:50

A / $\triangle$ admissible

σA is derivable

$$A \;/\; \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

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$A \mathrel{|\!\sim} \Delta$ admissible

σC is derivable for some $C \in \Delta$

$$\frac{\neg C \to A \vee B}{(\neg C \to A) \vee (\neg C \to B)}$$

$$\frac{A \vee B}{\{A, B\}}$$

Łukasiewicz 1952

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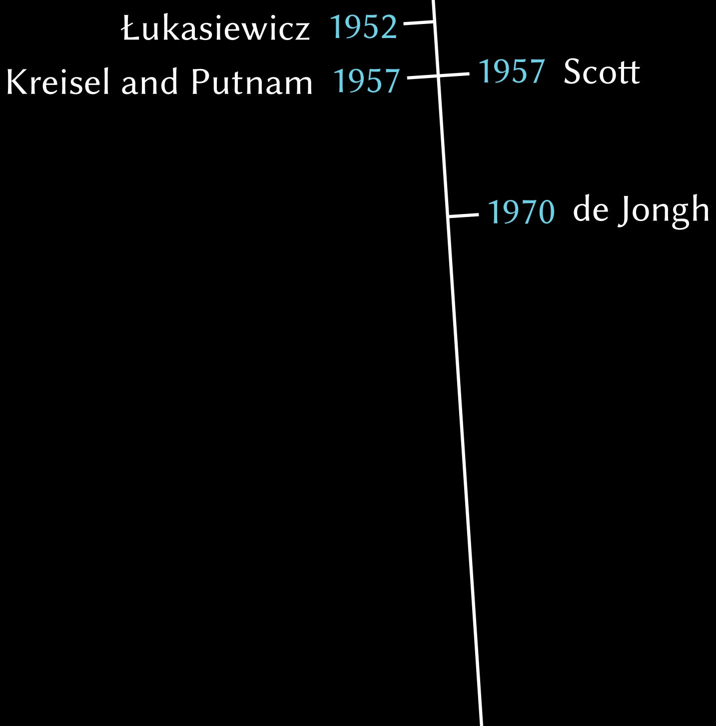
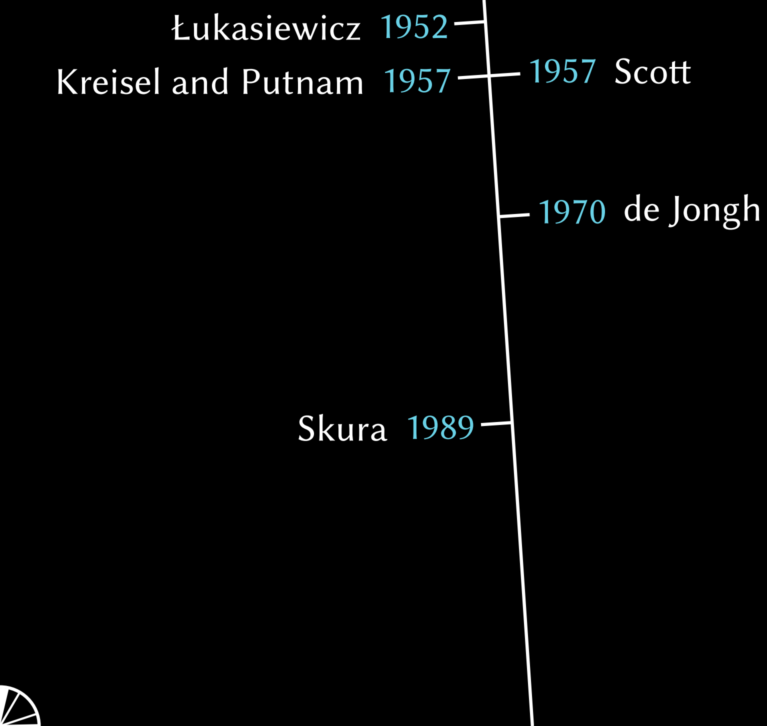
Kreisel and Putnam 1957


Łukasiewicz 1952
Kreisel and Putnam 1957
1957 Scott
1970 de Jongh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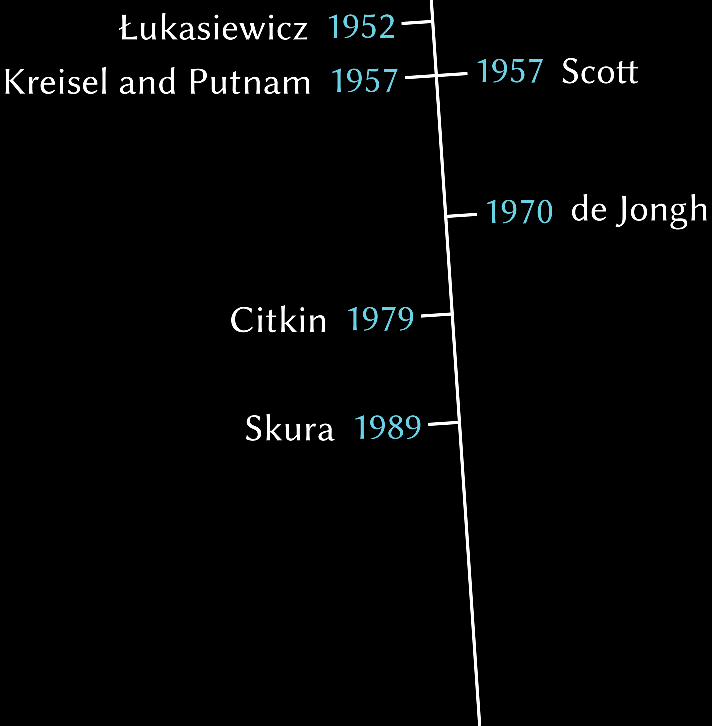
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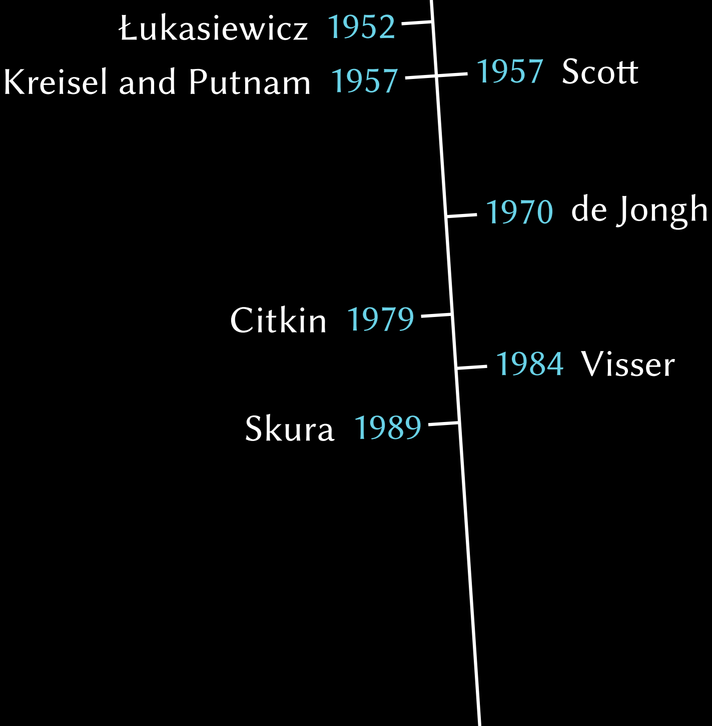
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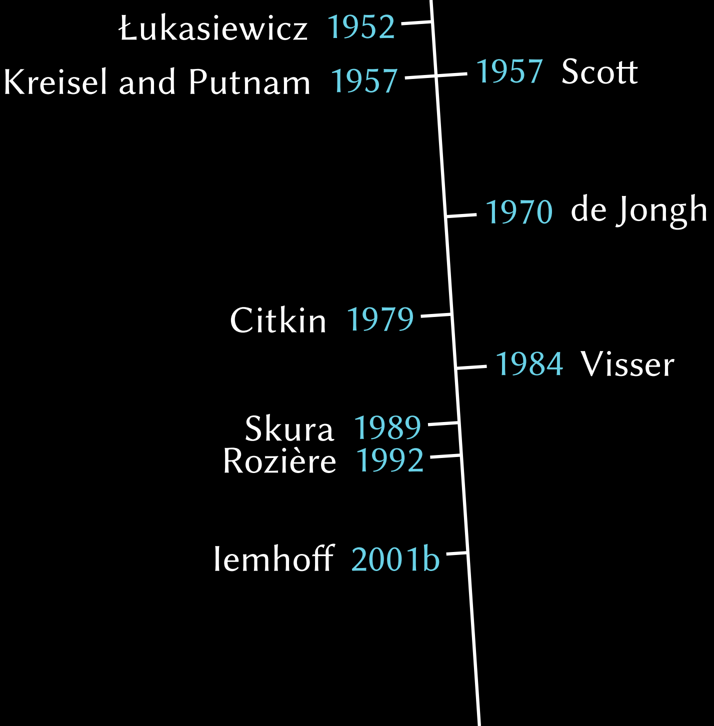
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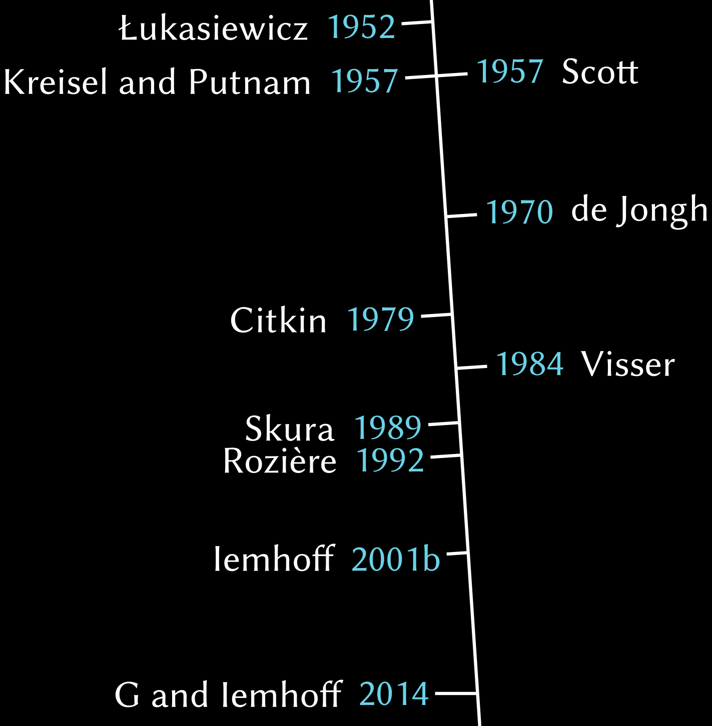
Rozière 1992
Iemhoff 2001b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G and Iemhoff 2014

Overview

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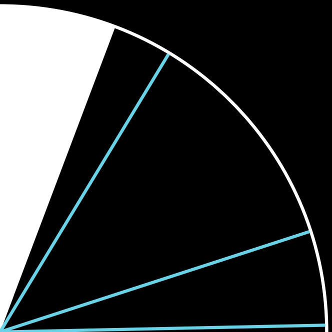

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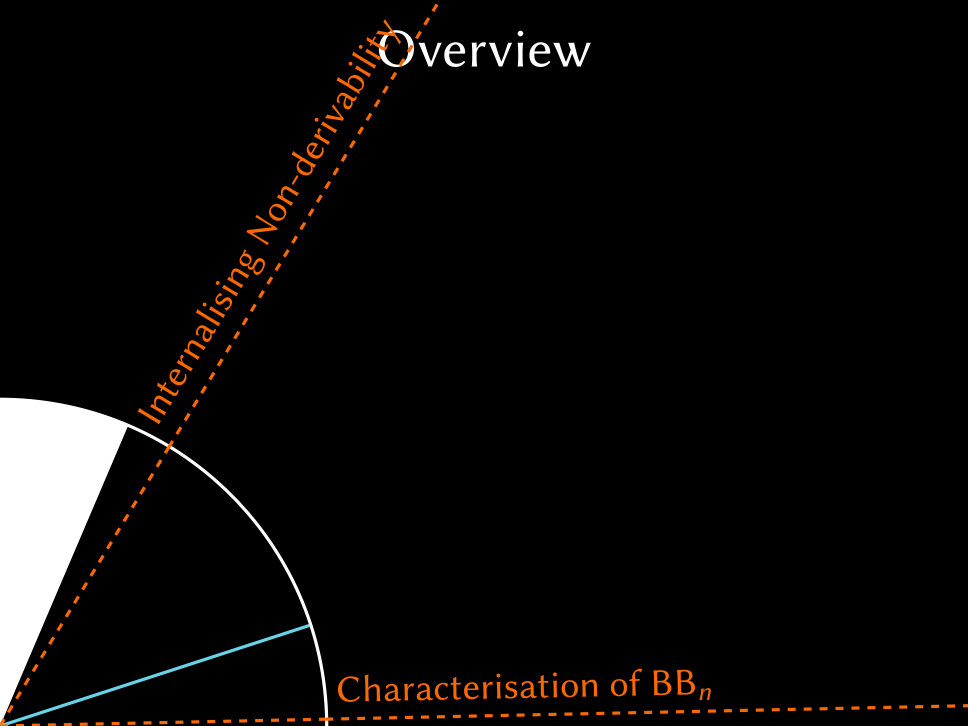

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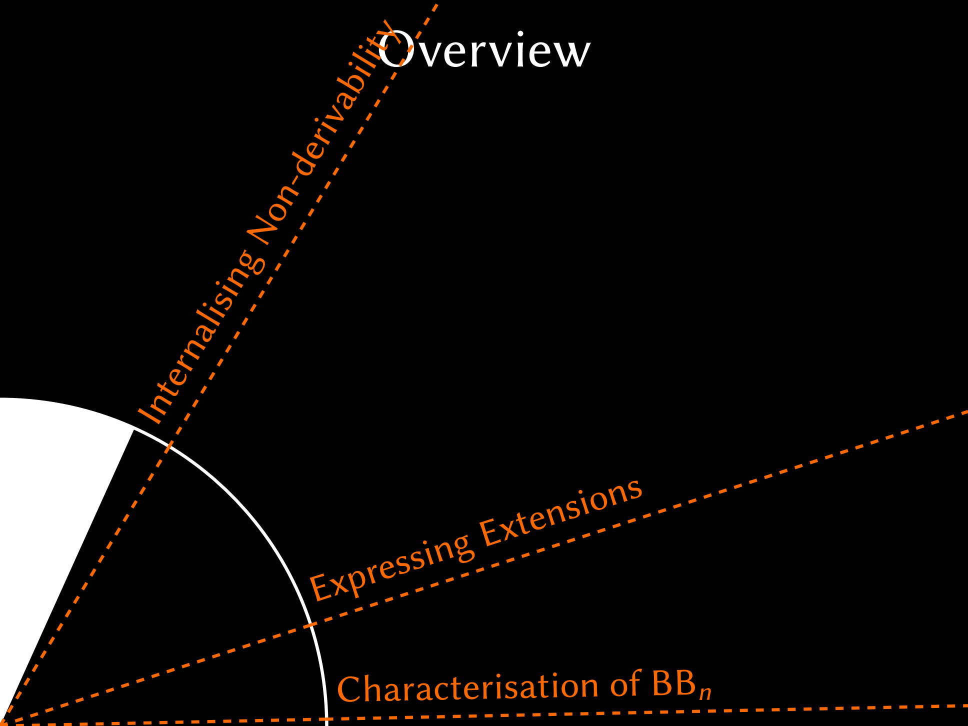
Internalising Non-derivability
Expressing Extensions
Characterisation of BB_n

Internalising Non-derivability

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Corollary:

There is no proper extension of IPC that inherits all its admissible rules.

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Suppose $\mathsf{L} \supsetneq \mathsf{IPC}$. This gives a $A \in \mathsf{L} - \mathsf{IPC}$. Hence there is a σ and a Δ with $\nvdash_{\mathsf{CPC}} C$ for all $C \in \Delta$ such that

$$\sigma A \mathrel{|\!\sim}_{\mathsf{IPC}} \Delta.$$

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But $\vdash_{\mathsf{L}} A$, so $\vdash_{\mathsf{L}} \sigma A$ holds as well. This yields some $C \in \Delta$ with such that $\vdash_{\mathsf{CPC}} C$, a contradiction.

Corollary (Iemhoff, 2001a):
IPC is the maximal intermediate logic with the rules below, for all *n*.

$$\frac{(\bigvee_{i=1}^{n} C_i \to A) \to \bigvee_{j=1}^{n} C_j}{\{(\bigvee_{i=1}^{n} C_i \to A) \to \bigvee C_j\}_{j=1}^{n}}$$

The universal model is the "smallest" model into which every finite model fits.

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1957 Rieger

1960 Nishimura

1968 de Jongh

1973 Urquhart

1975 Esakia and Grigolia

1978 Shehtman

1986 Bellissima

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$\mathsf{U}(X) \Vdash A$ iff $\vdash A$ for all $A \in \mathcal{L}(X)$.

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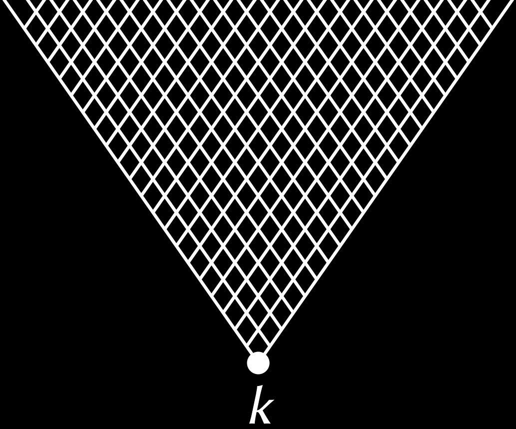
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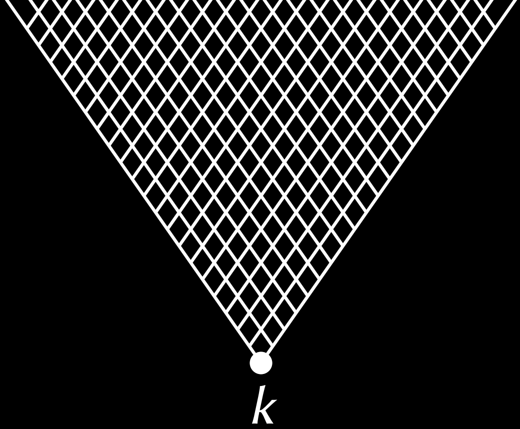

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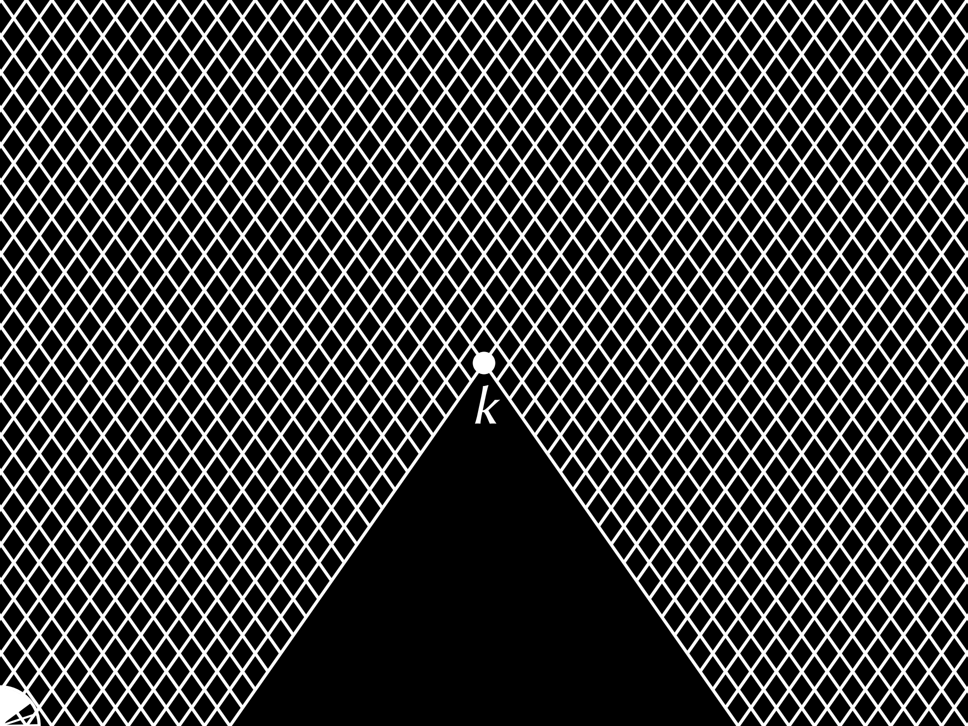
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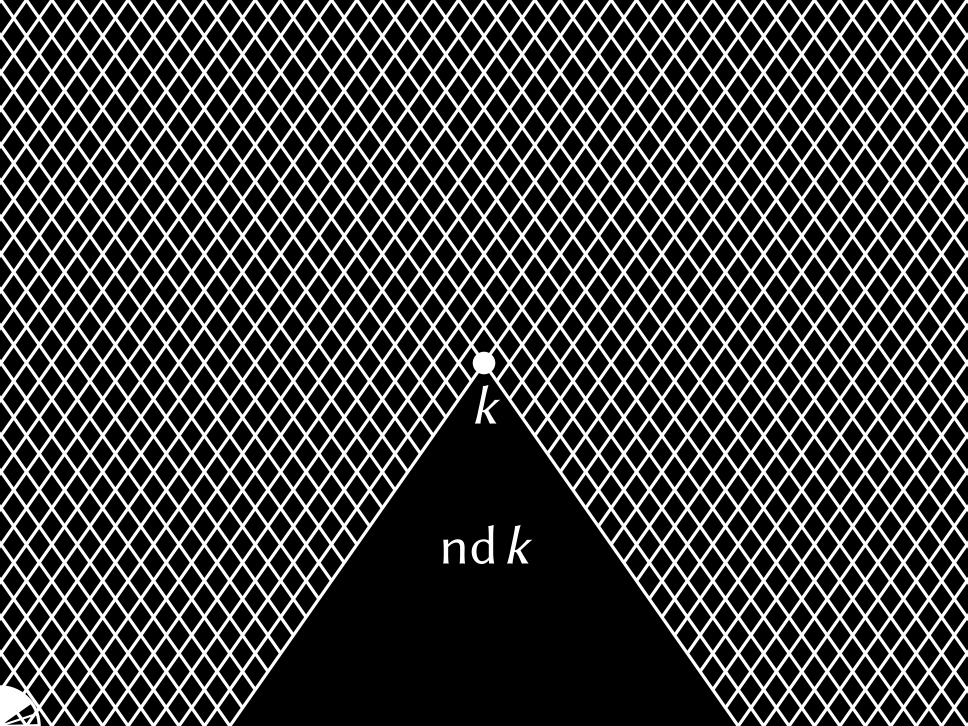
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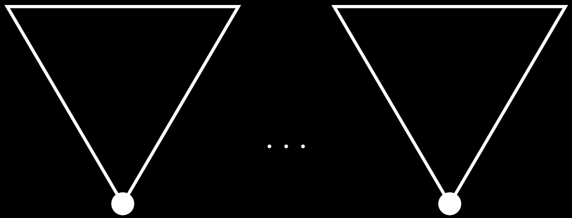
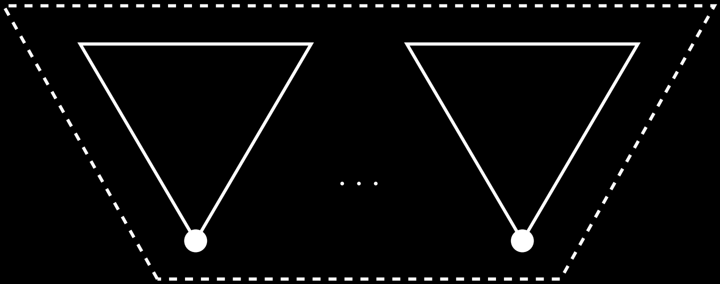
Expressing Extensions

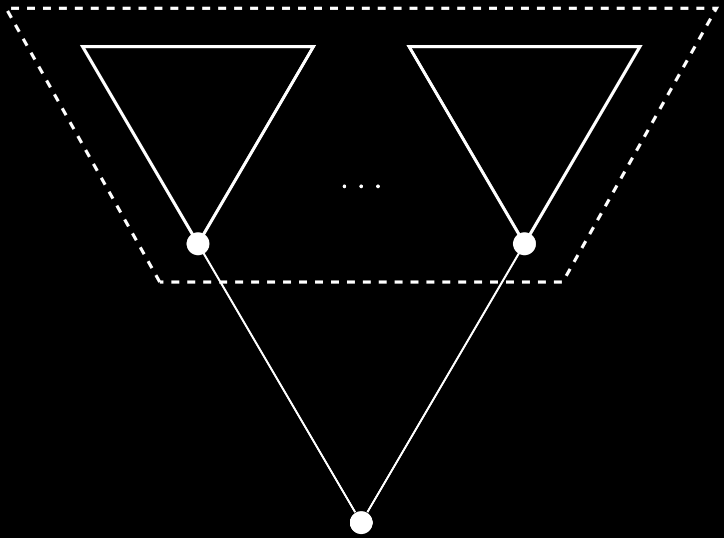
Extension Property



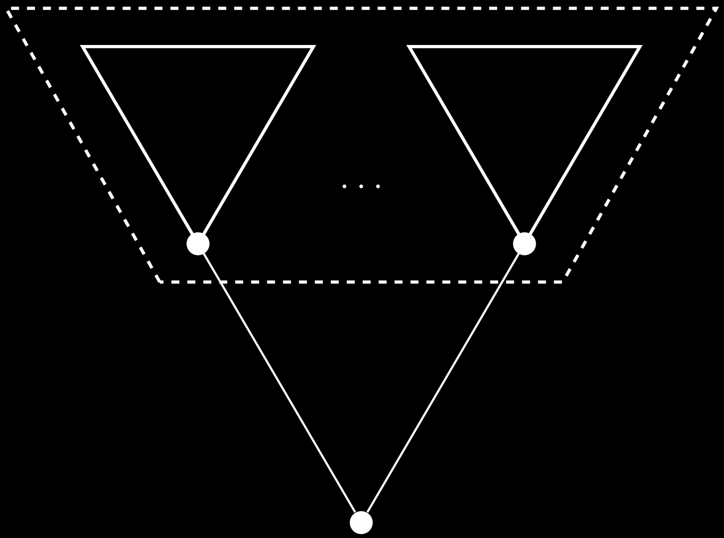
Extension Property



Extension Property



n^{th} Extension Property



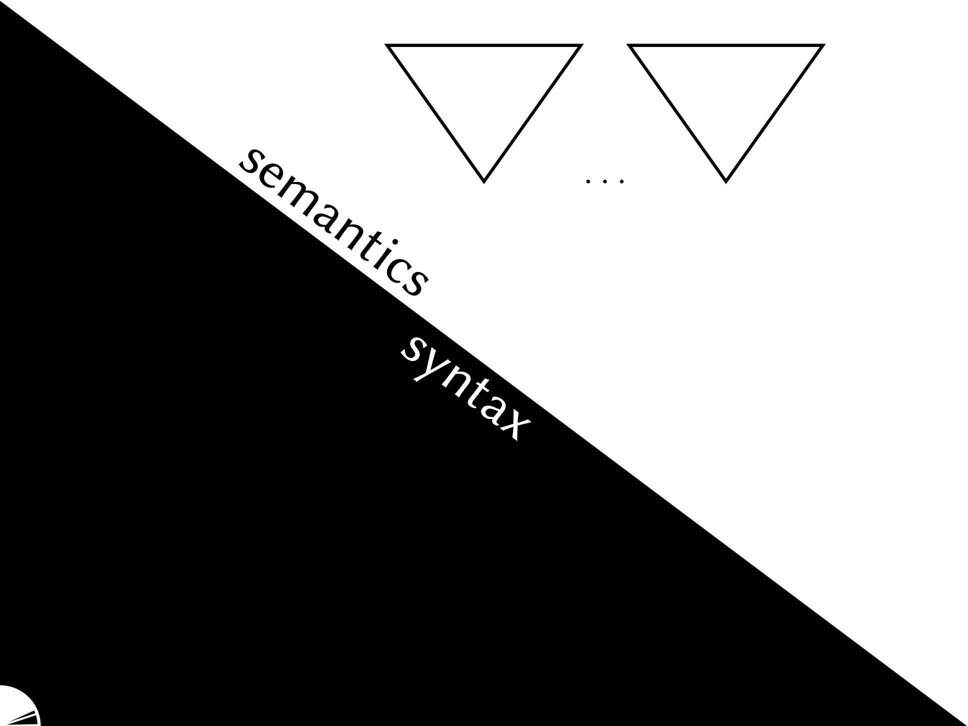
Visser Rules

$$\frac{(\bigvee_{i=1}^{n} C_i \to A) \to \bigvee_{j=1}^{n} C}{\{(\bigvee_{i=1}^{n} C_i \to A) \to C_j\}_{j=1}^{n}}$$

semantics

semantics

syntax



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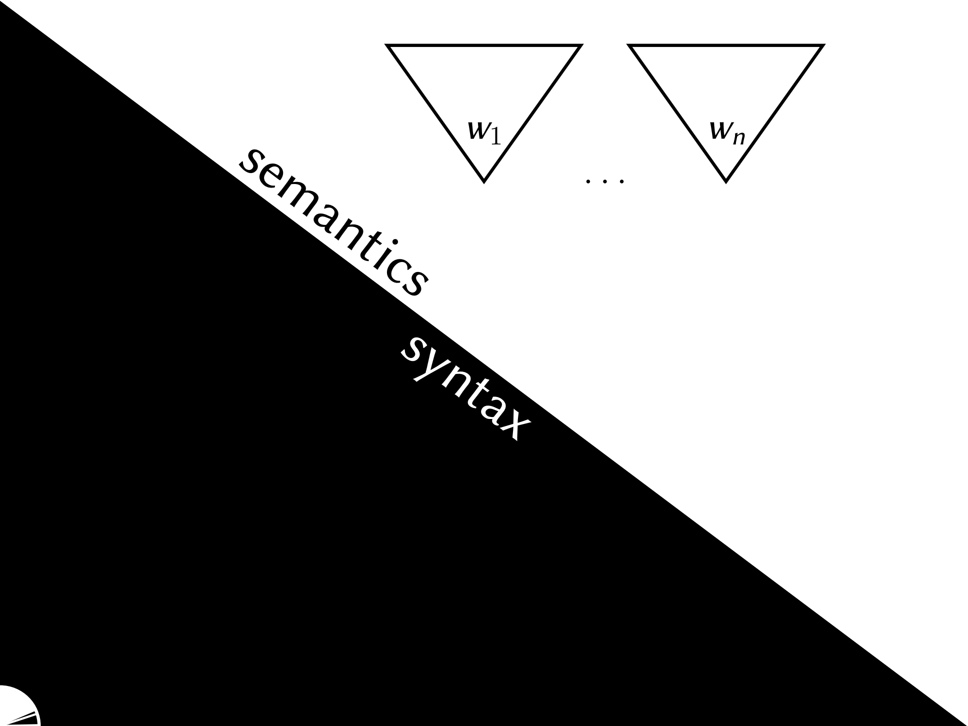
w_1
$\dots$
w_n
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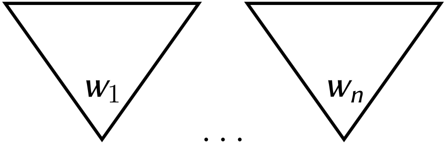
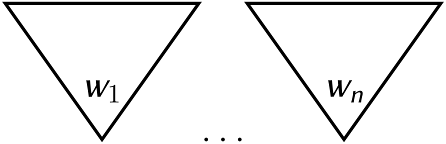


semantics

syntax

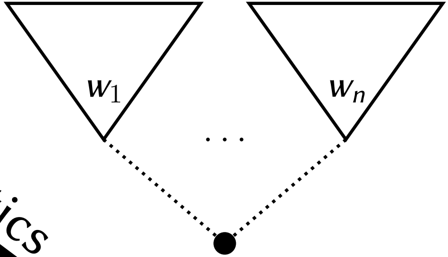
$$\left\{ \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee\nolimits_{i=1}^{n} \mathsf{up}\, w_i \right) \to \mathsf{nd}\, w_j \right\}_{j=1}^{n}$$



semantics

syntax

$$\dfrac{\left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \bigvee_{i=1}^{n} \mathsf{nd}\, w_i}{\left\{ \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \mathsf{nd}\, w_j \right\}_{j=1}^{n}}$$



semantics

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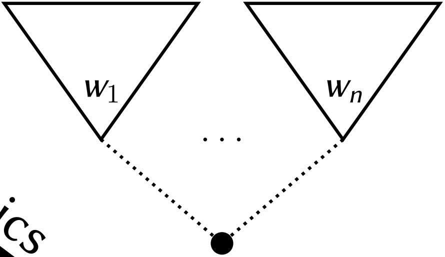

semantics

syntax

$$\frac{\left(\bigvee_{i=1}^{n} C_i \rightarrow A\right) \rightarrow \bigvee_{i=1}^{n} C_j}{\left\{ \left(\bigvee_{i=1}^{n} C_i \rightarrow A\right) \rightarrow C_j \right\}_{j=1}^{n}}$$

A logic with the
finite model property admits
the Visser rules iff it has
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A logic with the
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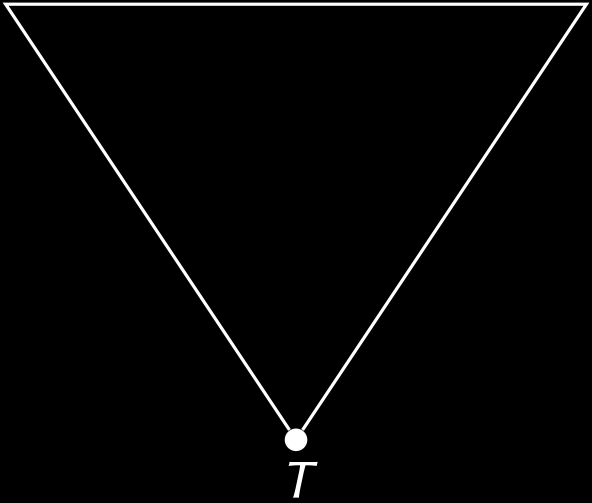
the Visser rules up to *n* iff it has
the extension property up to *n*.

Characterisation of BB_n

$$\mathrm{BB}_n = \mathrm{IPC} + \bigwedge_{i=0}^{n} \left(\left(x_i \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{j \neq i} x_j \right) \to \bigvee_{i=0}^{n} x_i$$

$\mathrm{BB}_n \nvdash A$ iff there is a
finite, proper and at most
n-fold branching tree T with $T \nVdash A$.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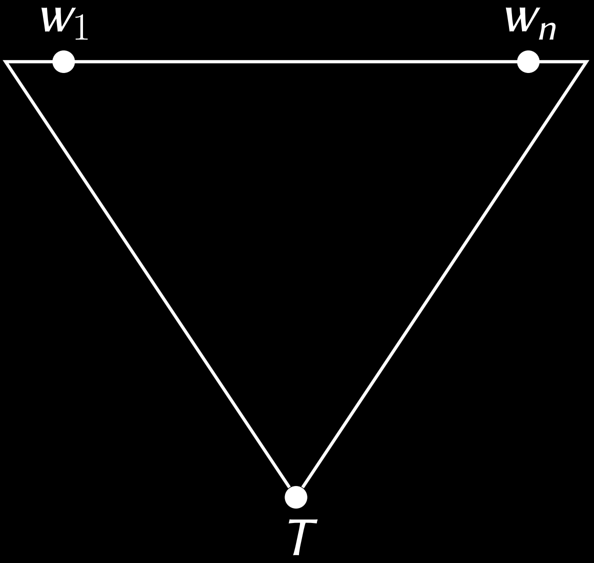
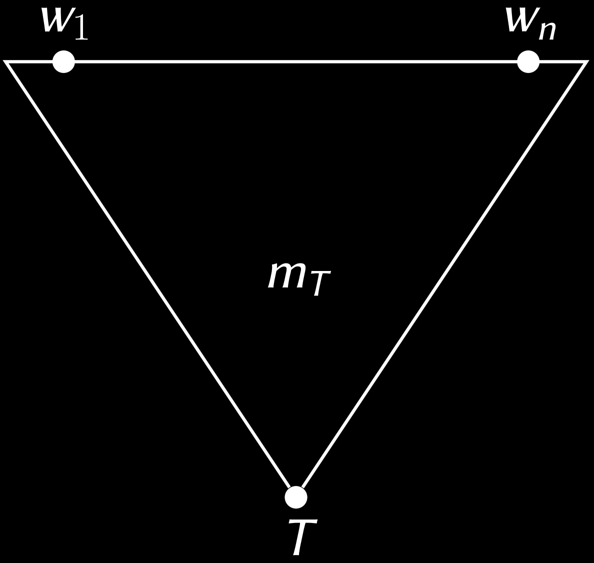


w_1
w_n
T


w_1
w_n
m_T
T

Theorem (G 2014):

If T is a finite, proper, and at most n-fold branching tree, then $\mathrm{nd}\, m_T \mathrel{|\!\sim} \{\mathrm{nd}\, w \mid w \in T \text{ maximal}\}$.

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Suppose $\nvdash_{\mathrm{BB}_n} A$. Then there is a finite, proper, and at most n-fold branching tree T such that $T \nVdash A$.

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$$\sigma A \vdash_{\mathrm{BB}_n} \text{nd}\, m_T$$

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$$\sigma A \vdash_{\mathrm{BB}_n} \mathrm{nd}\, m_T \mathrel{|\!\sim} \{\mathrm{nd}\, w \mid w \in T \text{ maximal}\}\,.$$

Corollary (G 2014):
BB_n is the maximal intermediate logic with the rules below.

$$\frac{(\bigvee_{i=1}^{n} C_i \to A) \to \bigvee_{j=1}^{n} C_j}{\{(\bigvee_{i=1}^{n} C_i \to A) \to \bigvee C_j\}_{j=1}^{n}}$$

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