Describing Admissible Rules

Jeroen P. Goudsmit
Utrecht University
April 28^{th} 2014

Disjunction Property

$$\frac{A \vee B \text{ derivable}}{A \text{ derivable or } B \text{ derivable}}$$

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

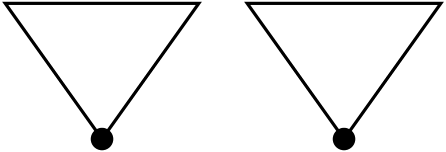
syntax

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

semantics

syntax

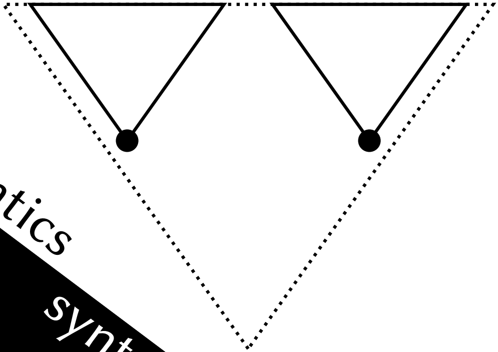
$$\frac{\vdash A \lor B}{\vdash A \text{ or } \vdash B}$$



semantics

syntax

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$



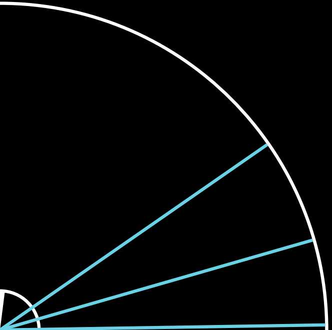
semantics

syntax

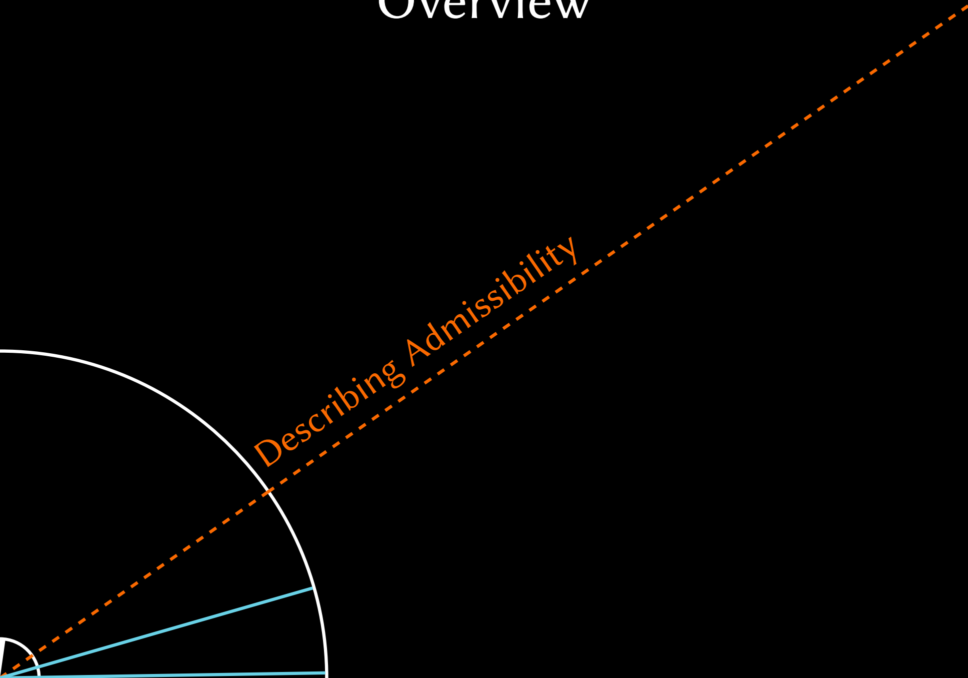
$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

Overview

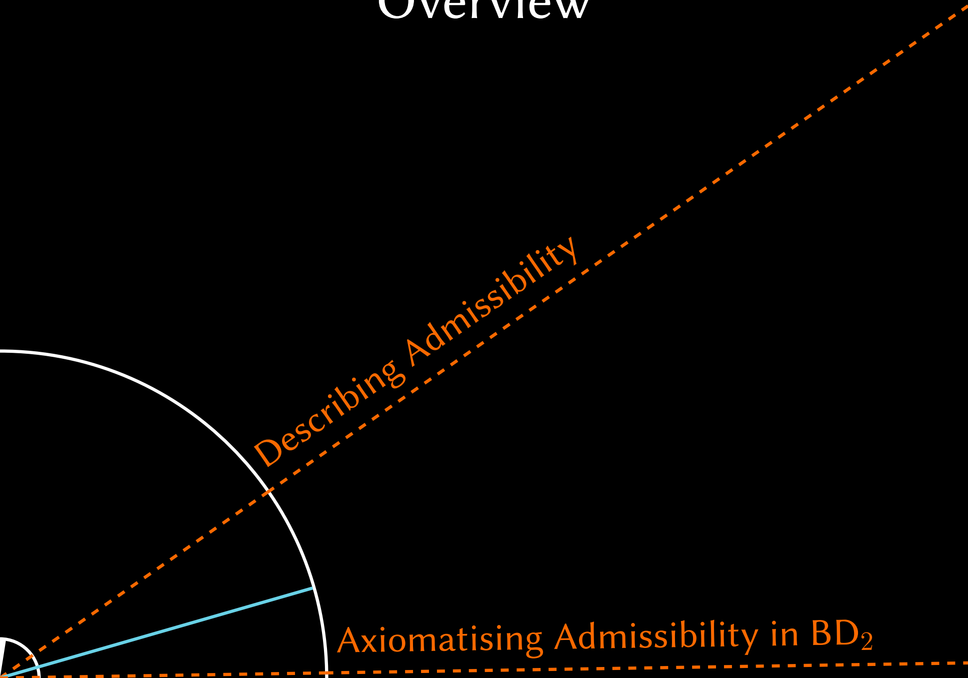
Overview



Overview

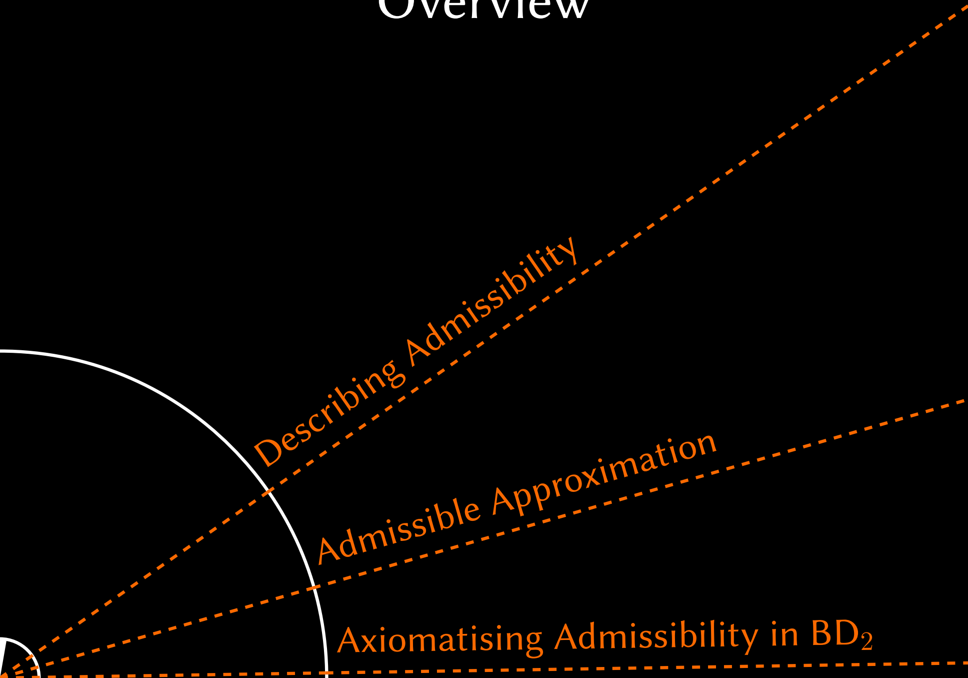


Overview



Describing Admissibility
Axiomatising Admissibility in BD_2

Overview



Describing Admissibility
Admissible Approximation
Axiomatising Admissibility in BD2

$A \;/\; \triangle$ admissible

σA is derivable

$A \;/\; \Delta$ admissible

σC is derivable for some $C \in \Delta$

σA is derivable

$A \mathrel{|\!\sim} \Delta$ admissible

σC is derivable for some $C \in \Delta$

$$\frac{\neg C \to A \lor B}{(\neg C \to A) \lor (\neg C \to B)}$$

$$\frac{\neg C \to A \lor B}{\{\ \neg C \to A, \qquad \neg C \to B\ \}}$$

1974 Gabbay and de Jongh

1974 Gabbay and de Jongh

1986 Maksimova

1952 Łukasiewicz

1974 Gabbay and de Jongh

1986 Maksimova

1952 Łukasiewicz

1957 Kreisel and Putnam

1974 Gabbay and de Jongh

1986 Maksimova

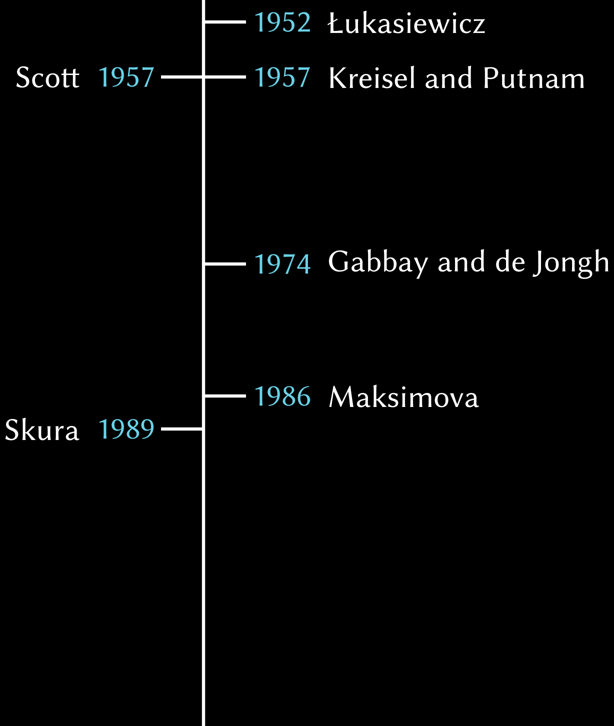
1952 Łukasiewicz

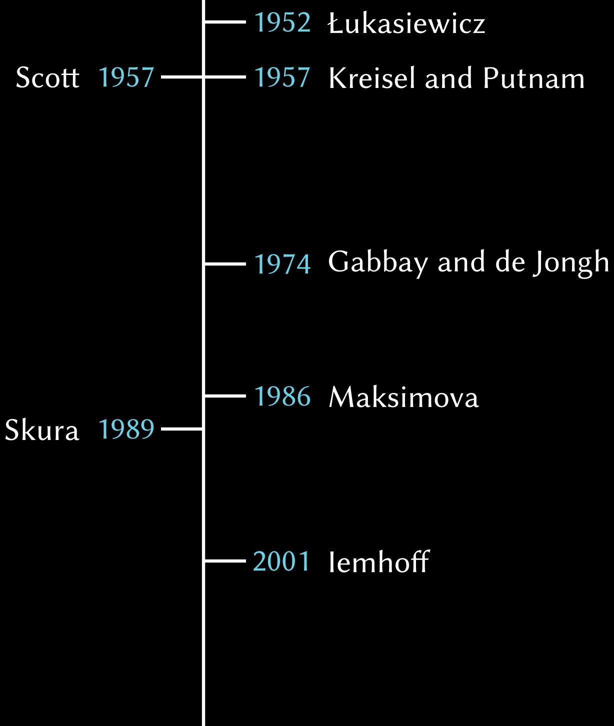
Scott 1957

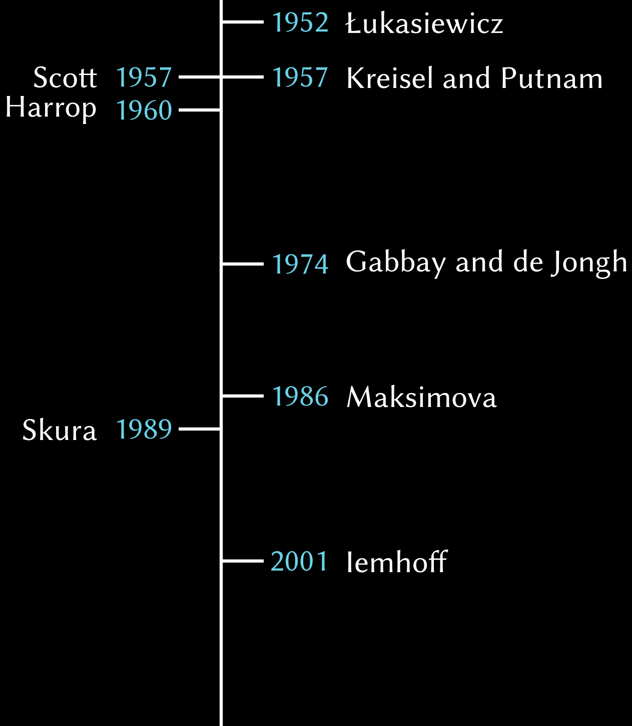
1957 Kreisel and Putnam

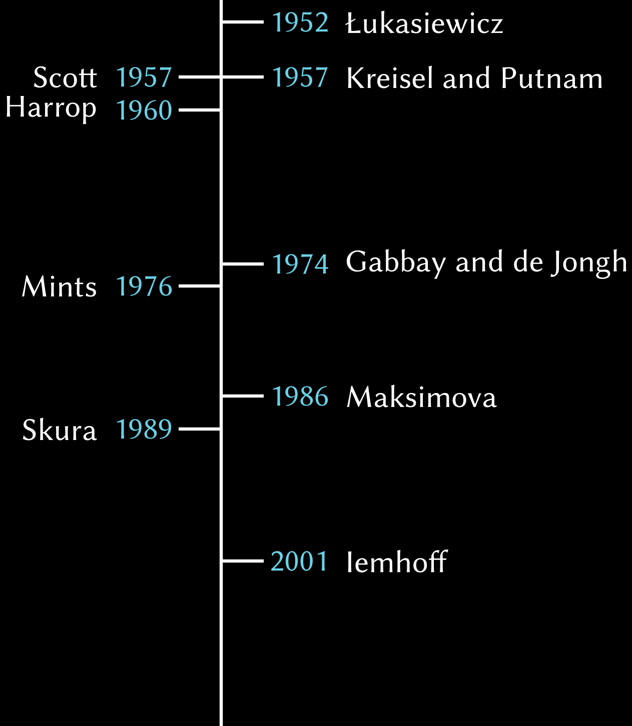
1974 Gabbay and de Jongh

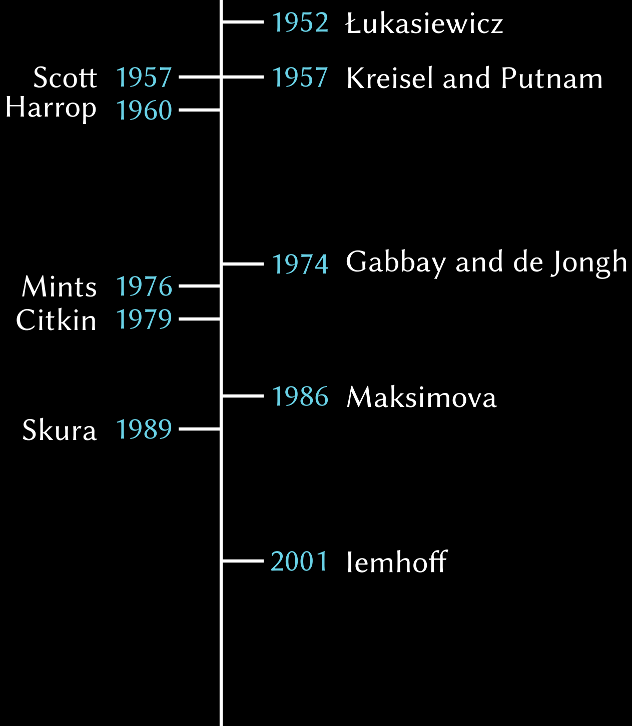
1986 Maksimova

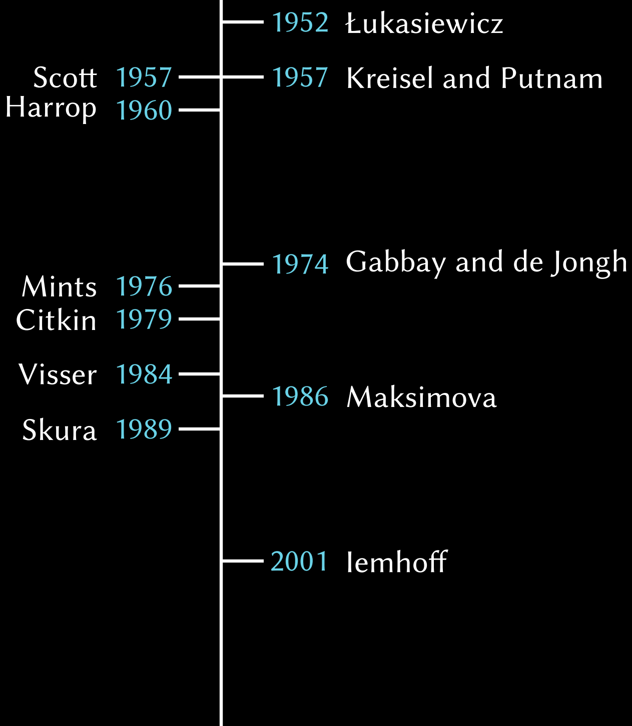

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
1974 Gabbay and de Jongh
1986 Maksimova
Skura 1989

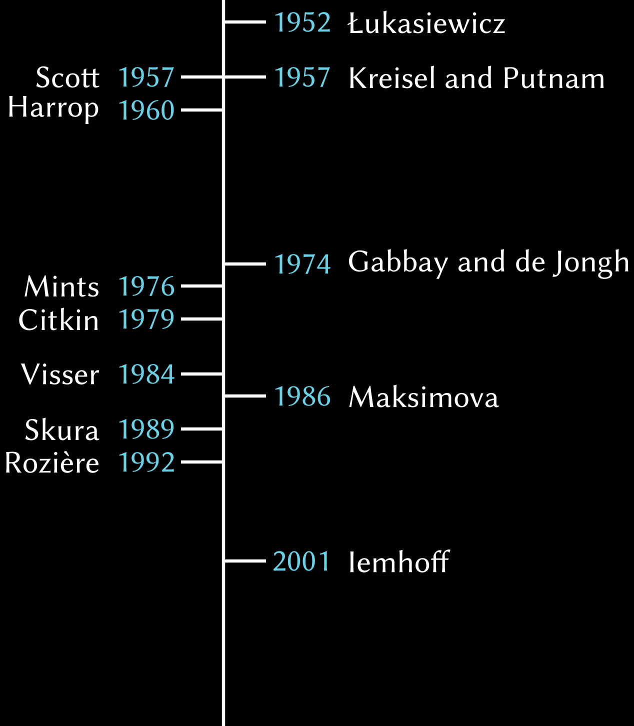

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
1974 Gabbay and de Jongh
1986 Maksimova
Skura 1989
2001 Iemhoff

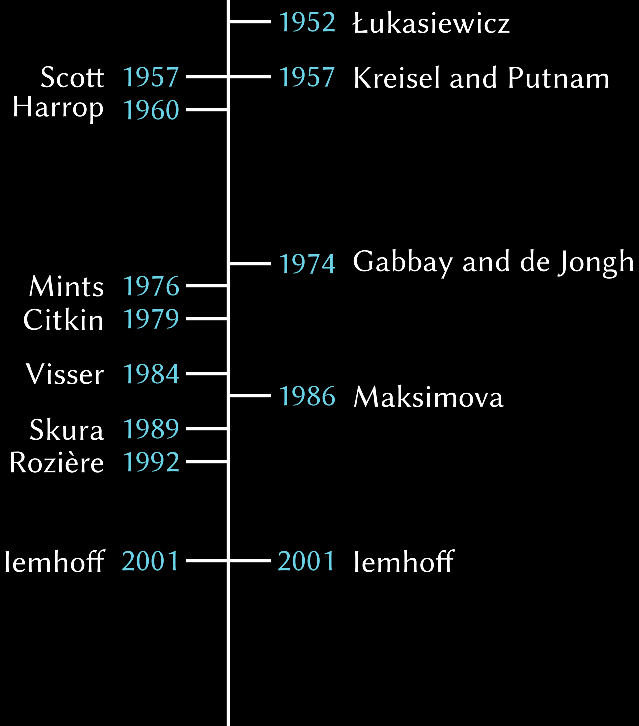

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
1986 Maksimova
Skura 1989
2001 Iemhoff

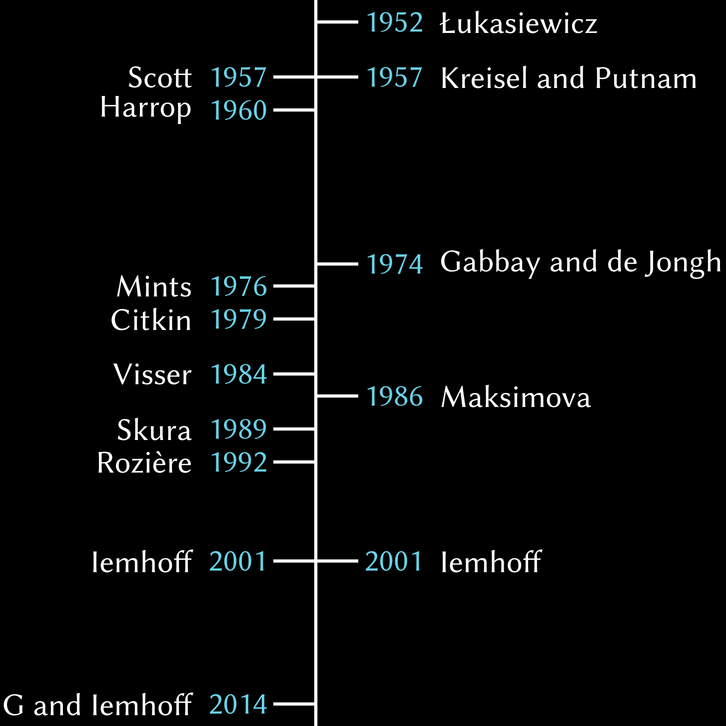

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
1986 Maksimova
Skura 1989
2001 Iemhoff

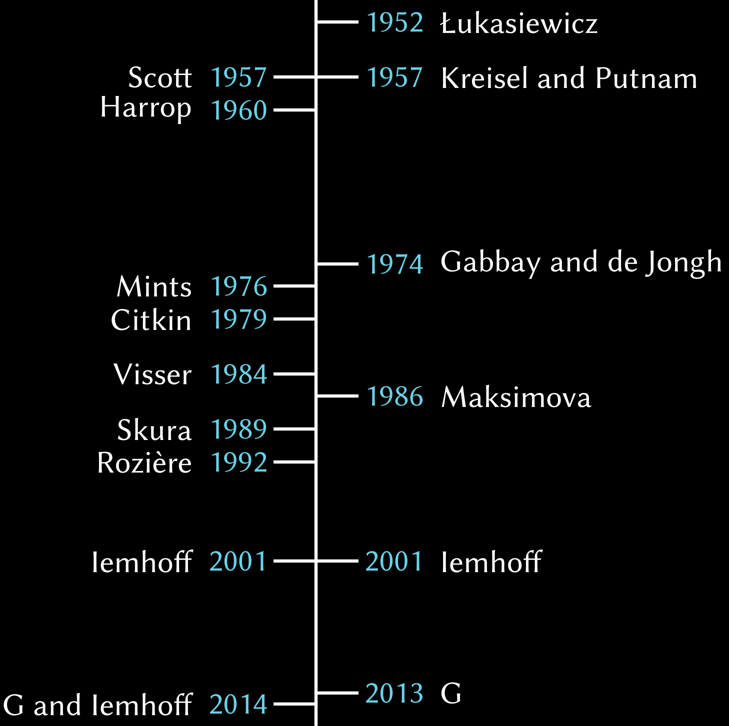

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
1986 Maksimova
Skura 1989
2001 Iemhoff


1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
Visser 1984
1986 Maksimova
Skura 1989
2001 Iemhoff


1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
Visser 1984
1986 Maksimova
Skura 1989
Rozière 1992
2001 Iemhoff


1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
Visser 1984
1986 Maksimova
Skura 1989
Rozière 1992
Iemhoff 2001
2001 Iemhoff


1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
Visser 1984
1986 Maksimova
Skura 1989
Rozière 1992
Iemhoff 2001
2001 Iemhoff
G and Iemhoff 2014

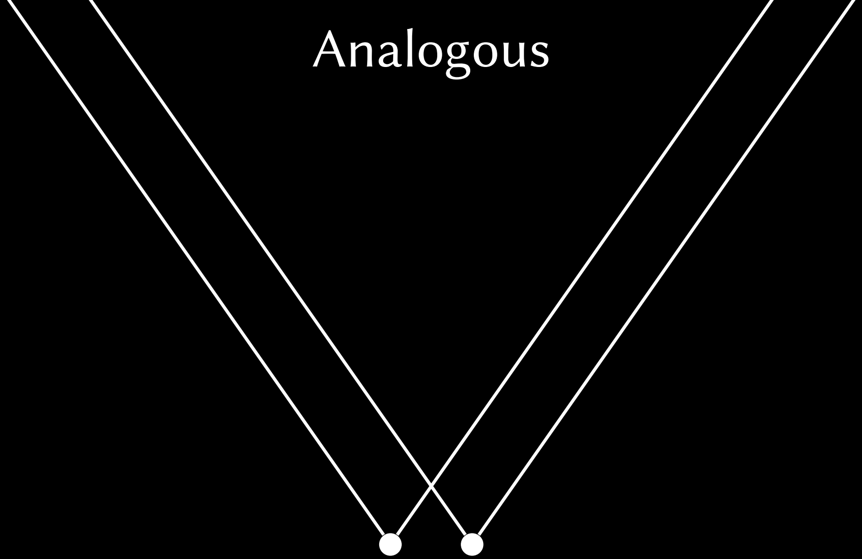

1952 Łukasiewicz
Scott 1957
1957 Kreisel and Putnam
Harrop 1960
1974 Gabbay and de Jongh
Mints 1976
Citkin 1979
Visser 1984
1986 Maksimova
Skura 1989
Rozière 1992
Iemhoff 2001
2001 Iemhoff
2013 G
G and Iemhoff 2014

Visser Rules

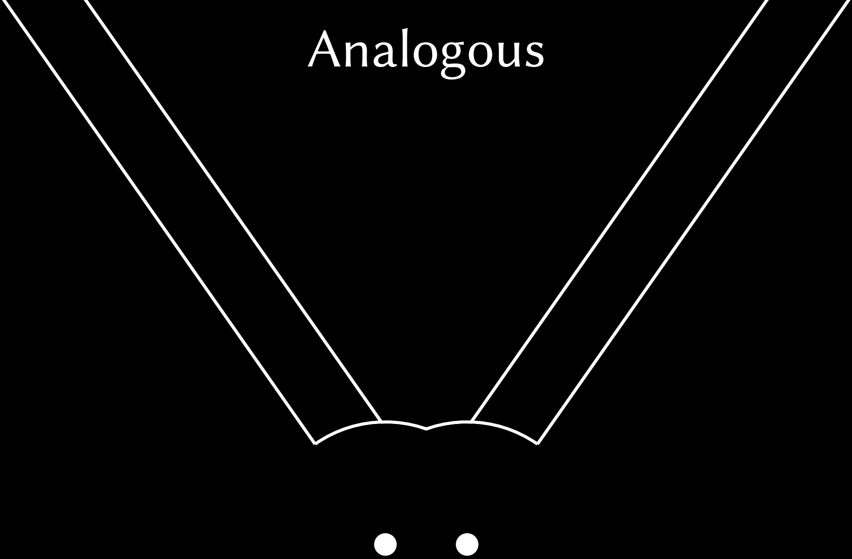
$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\bigvee \left\{ (\bigvee \Delta \to A) \to C \;\middle|\; C \in \Delta \right\}}$$

Analogous

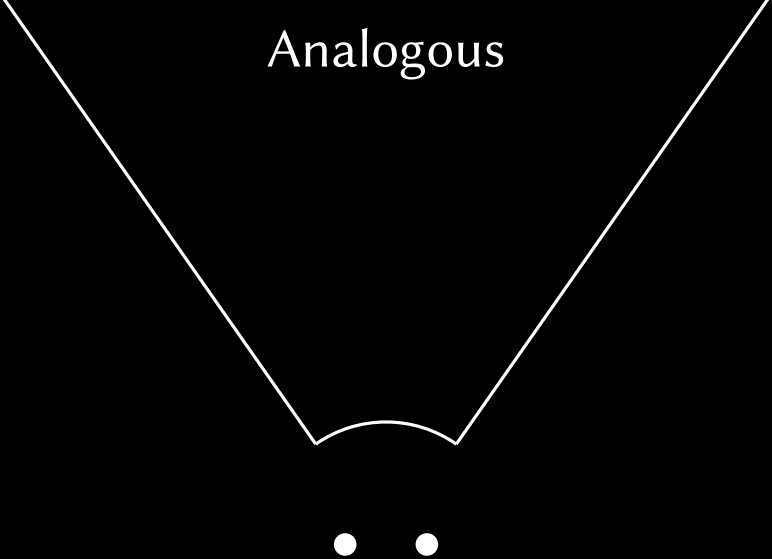
Analogous



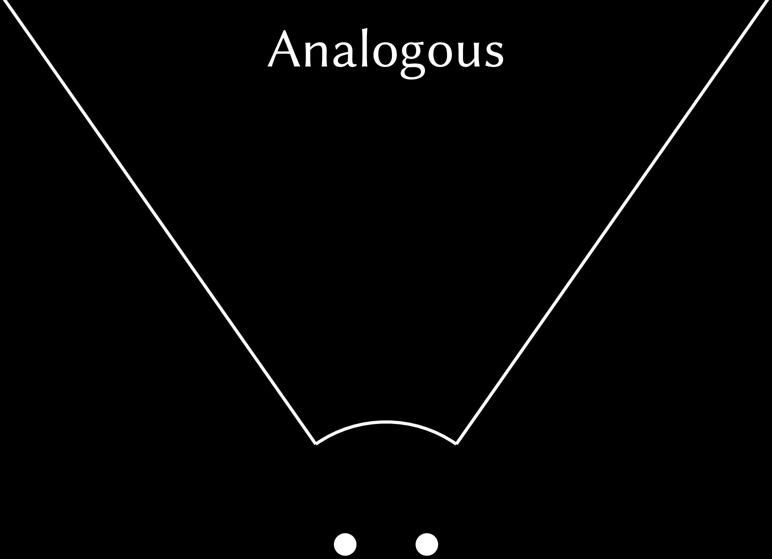
Analogous



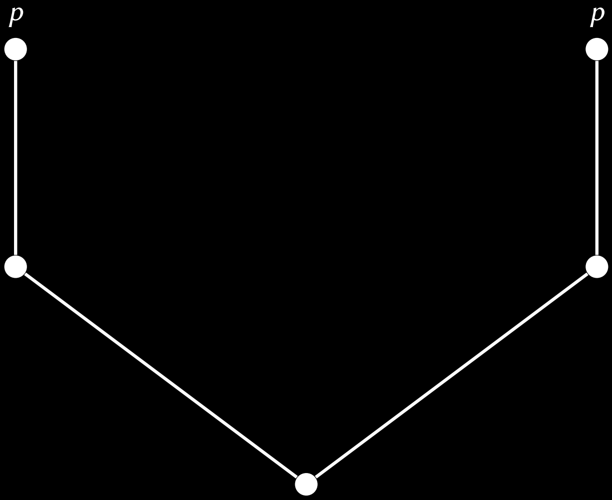
Analogous



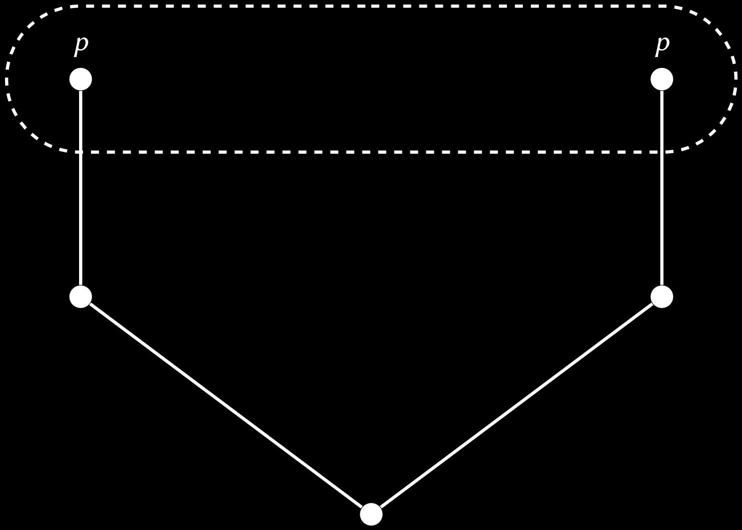
Analogous

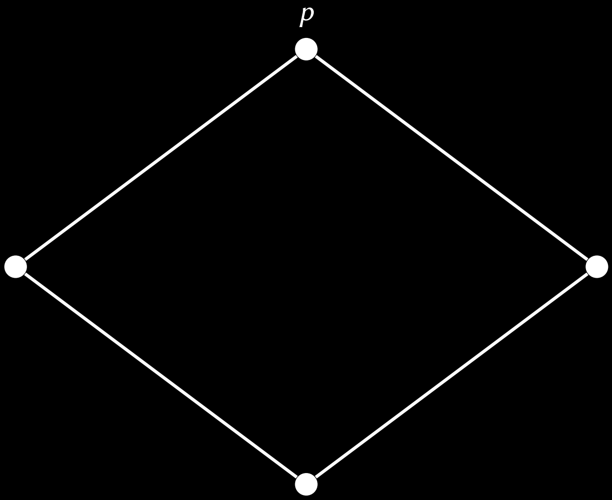


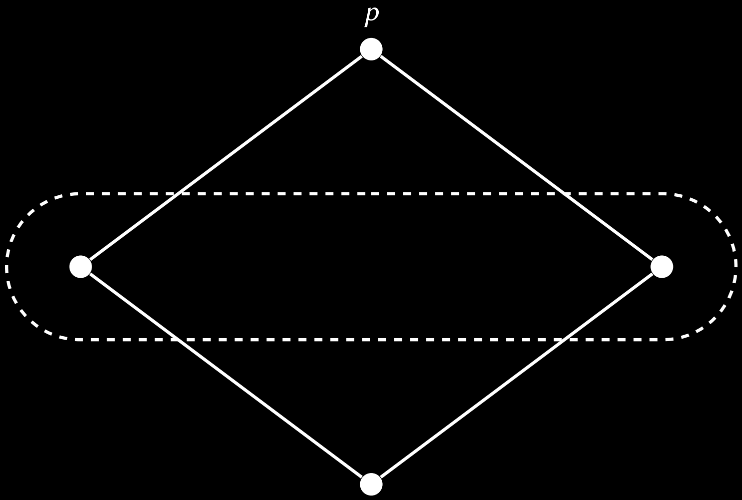
$k \equiv l$ when $v(k) = v(l)$ and $k \leq u$ iff $l \leq u$ for all $u \neq k, l$



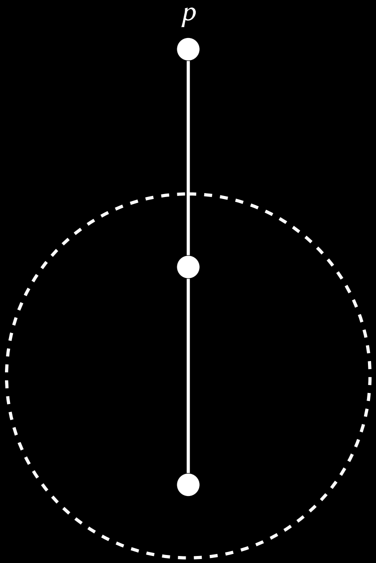
p
p


p
p


p


p


p


p

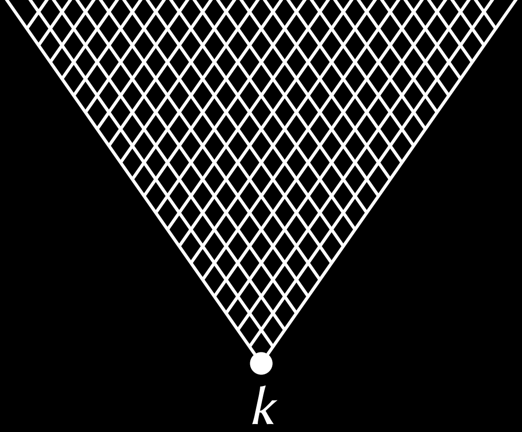

p

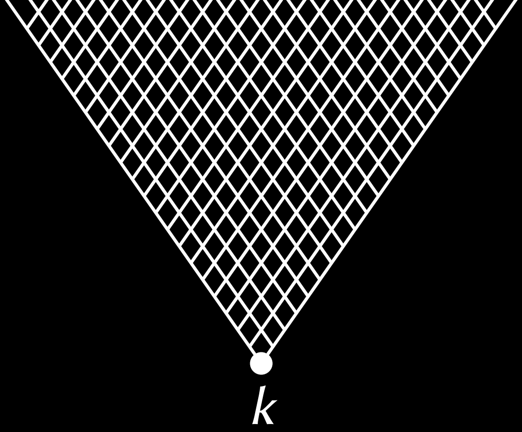
Jankov–de Jongh formulae

In suitable models have

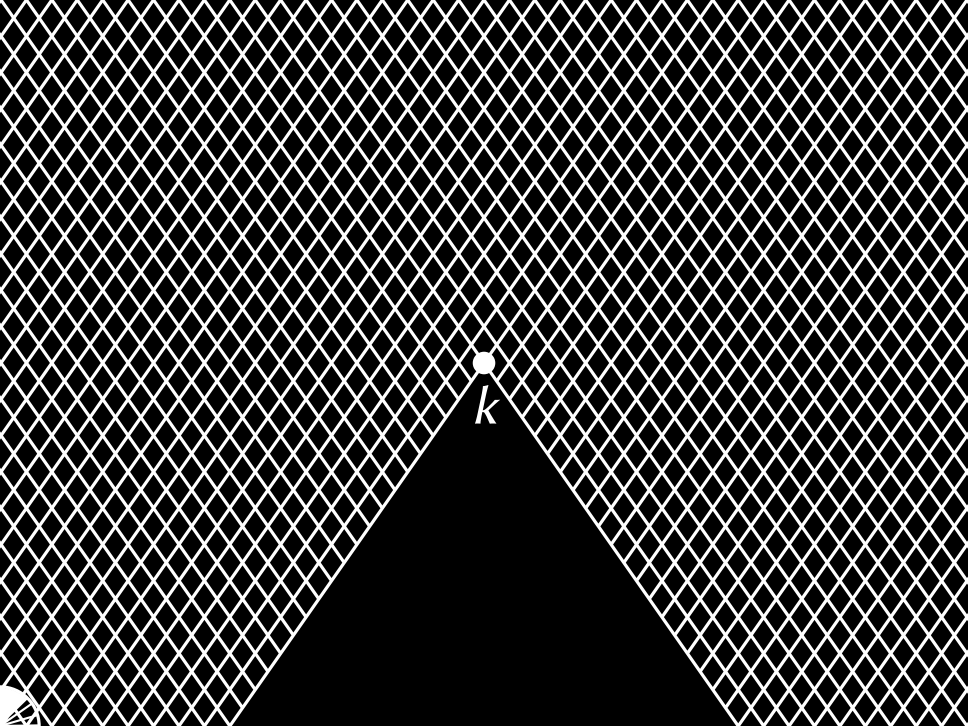
$$l \Vdash \mathrm{up}\, k \quad \text{iff} \quad k \le l$$
$$l \Vdash \mathrm{nd}\, k \quad \text{iff} \quad l \not\le k$$

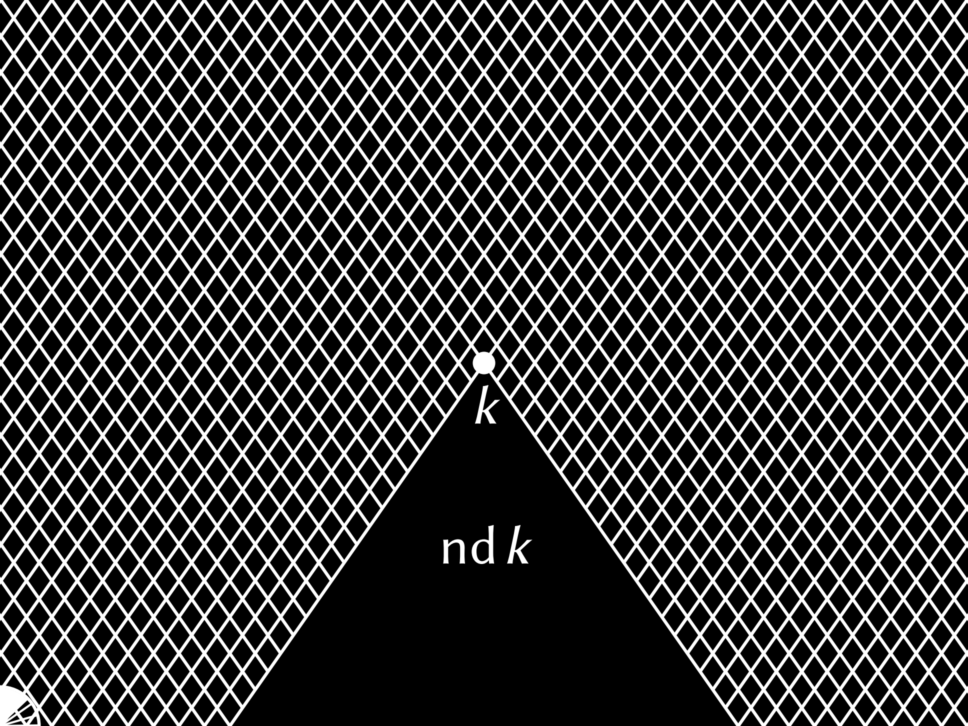

k

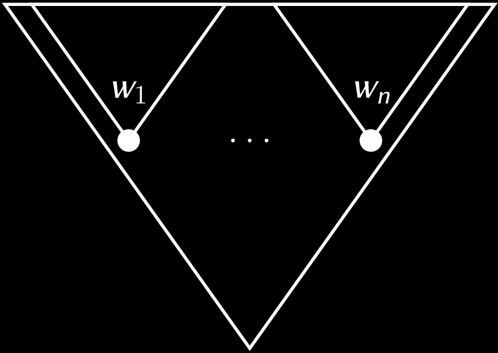

k



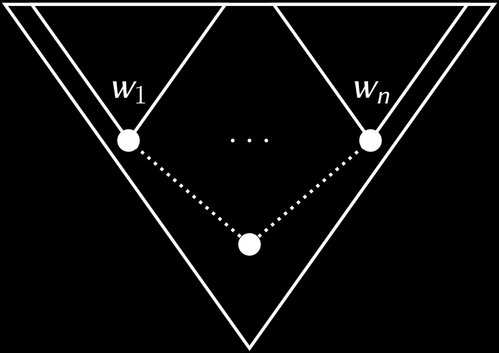
up k


k

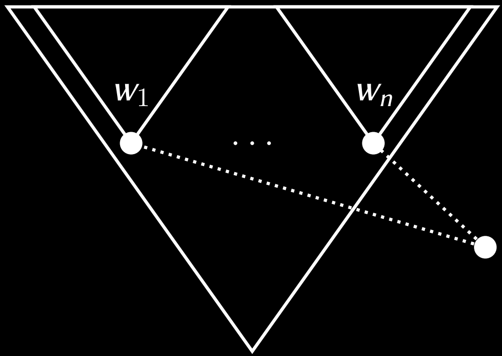

k
nd k



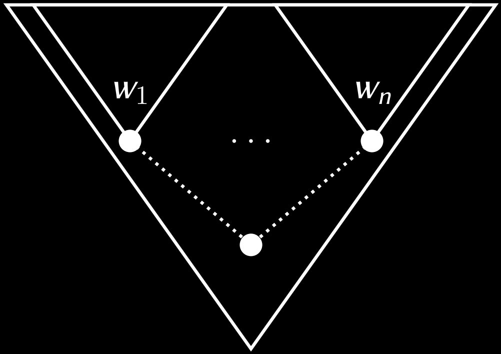
w_1
...
w_n



w_1
w_n
. . .

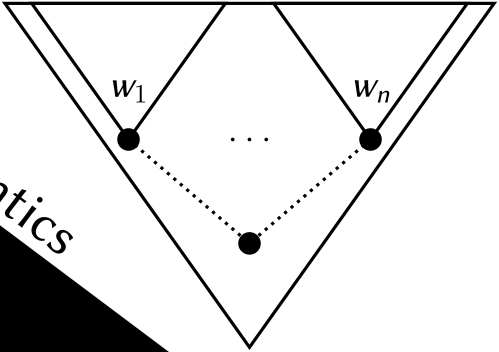


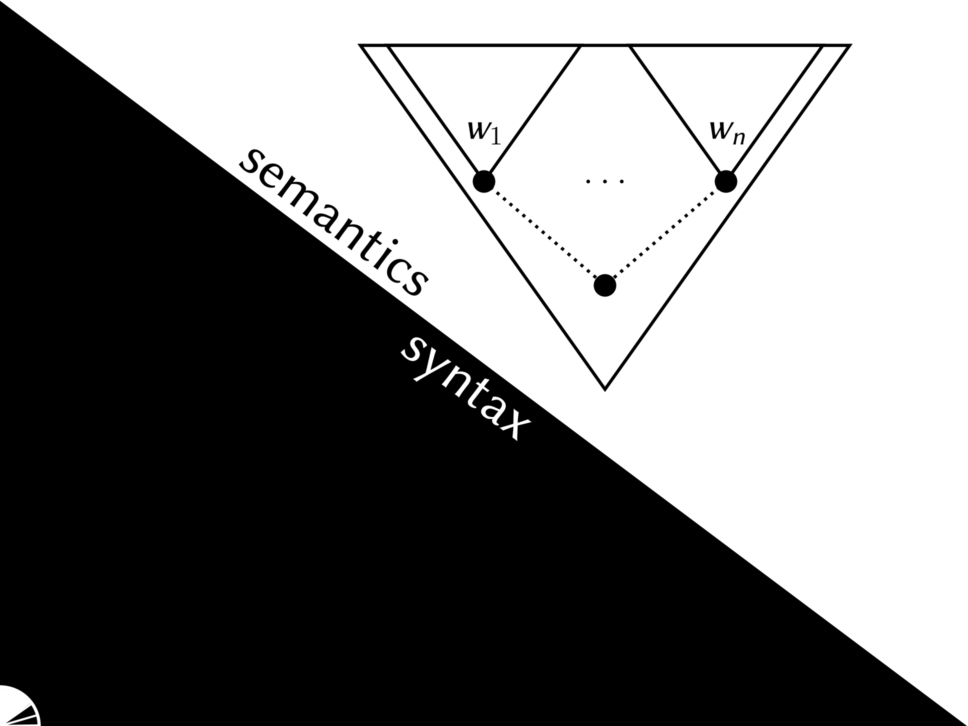
w_1
w_n

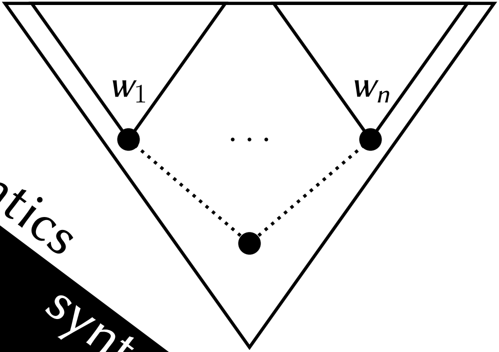


w_1
w_n
. . .

semantics



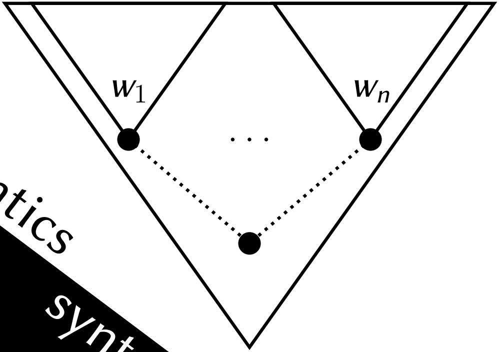

semantics
syntax
w_1
w_n
. . .



semantics

syntax

$$\frac{\left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \bigvee_{i=1}^{n} \mathsf{nd}\, w_i}{\bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \mathsf{nd}\, w_j}$$



semantics

syntax

$$\frac{\left(\bigvee \Delta \quad \to A \qquad \right) \to \bigvee \Delta}{\bigvee_{C \in \Delta} \left(\bigvee \Delta \quad \to A \qquad \right) \to C}$$

An axiomatisation of admissibility is a set of rules R with

$$\vdash_R = \mid\!\sim$$

Logic of Depth n

$$
\begin{aligned}
\mathrm{bd}_0 &= \bot \\
\mathrm{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathrm{bd}_n).
\end{aligned}
$$

Logic of Depth n

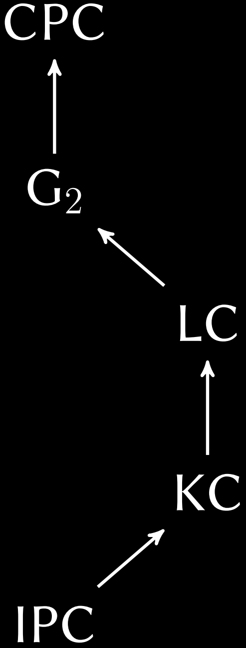
$$\begin{aligned} \mathrm{bd}_0 &= \bot \\ \mathrm{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathrm{bd}_n). \end{aligned}$$

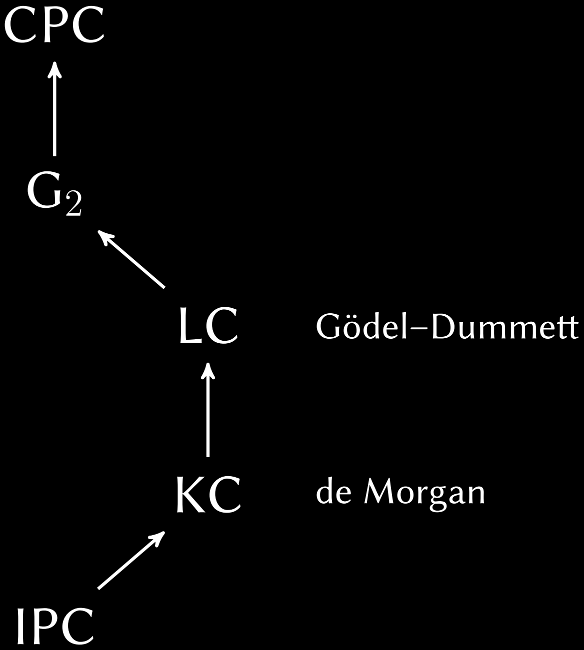
$$\mathrm{BD}_n = \mathrm{IPC} + \mathrm{bd}_n$$

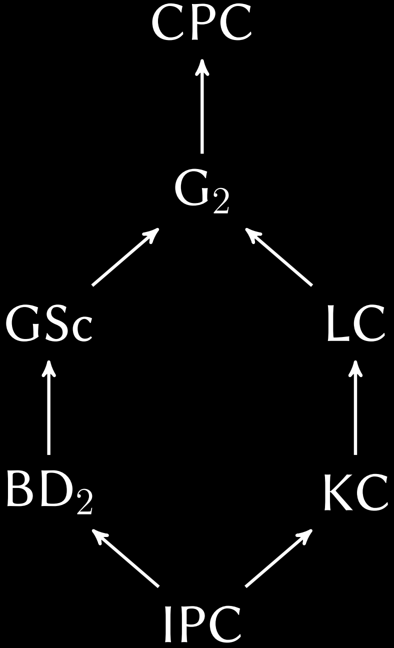
CPC

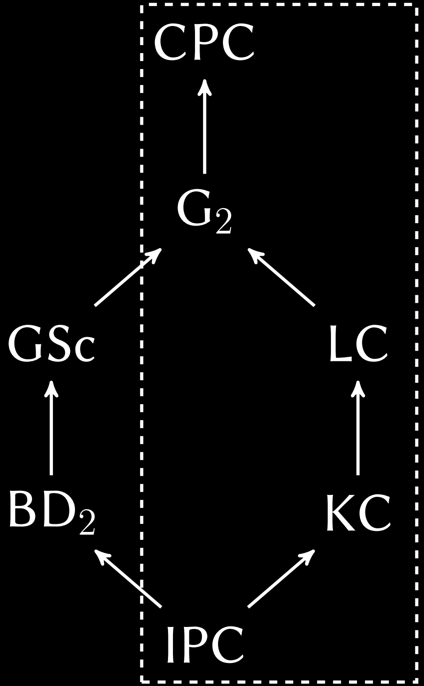
↑

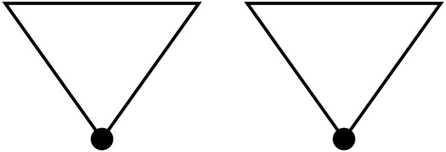
IPC

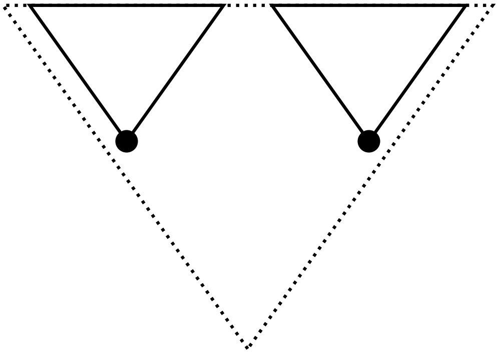

CPC
G_2
LC
KC
IPC

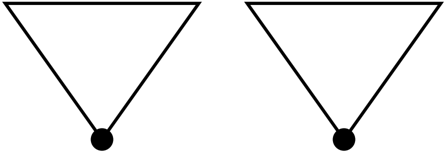

CPC
G_2
LC
Gödel–Dummett
KC
de Morgan
IPC

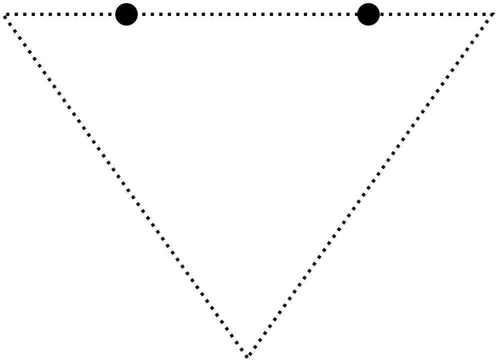

CPC
G2
GSc
LC
BD2
KC
IPC

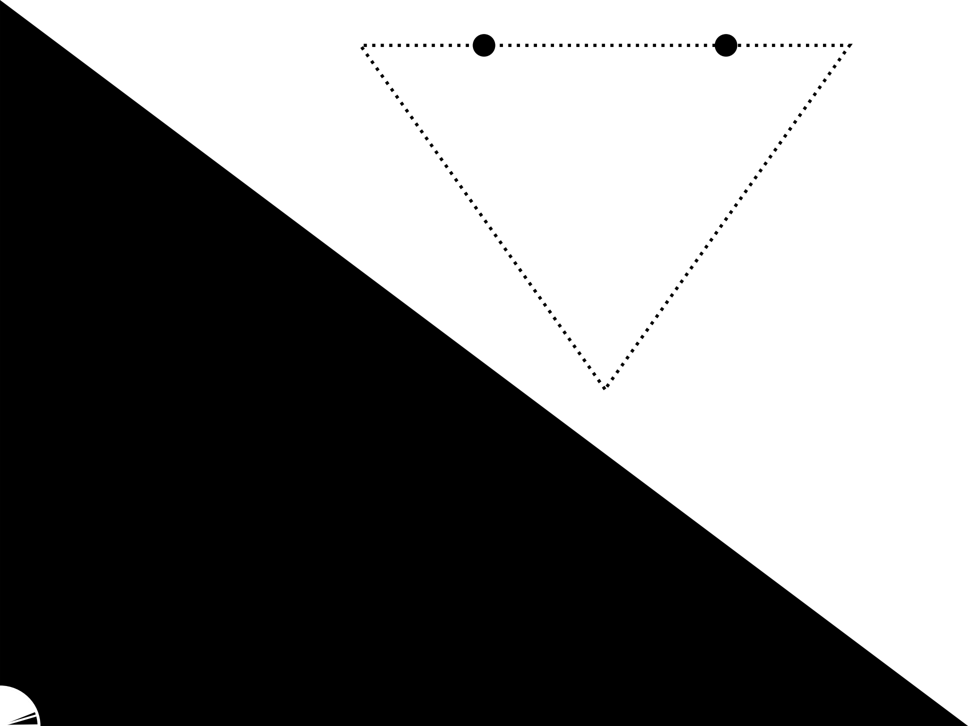

CPC
G2
GSc
LC
BD2
KC
IPC

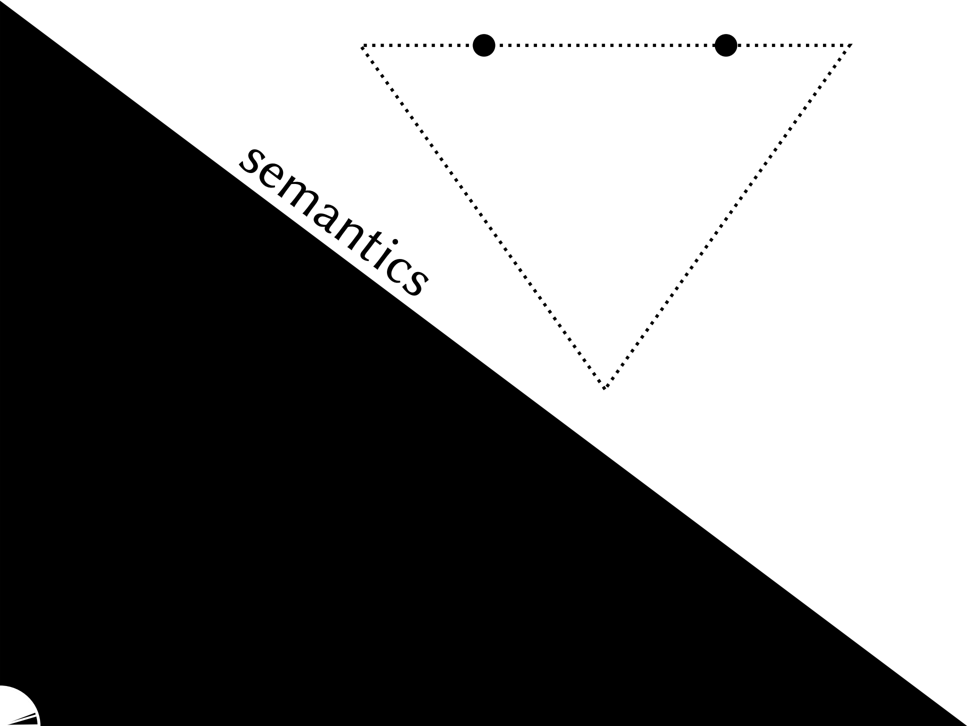


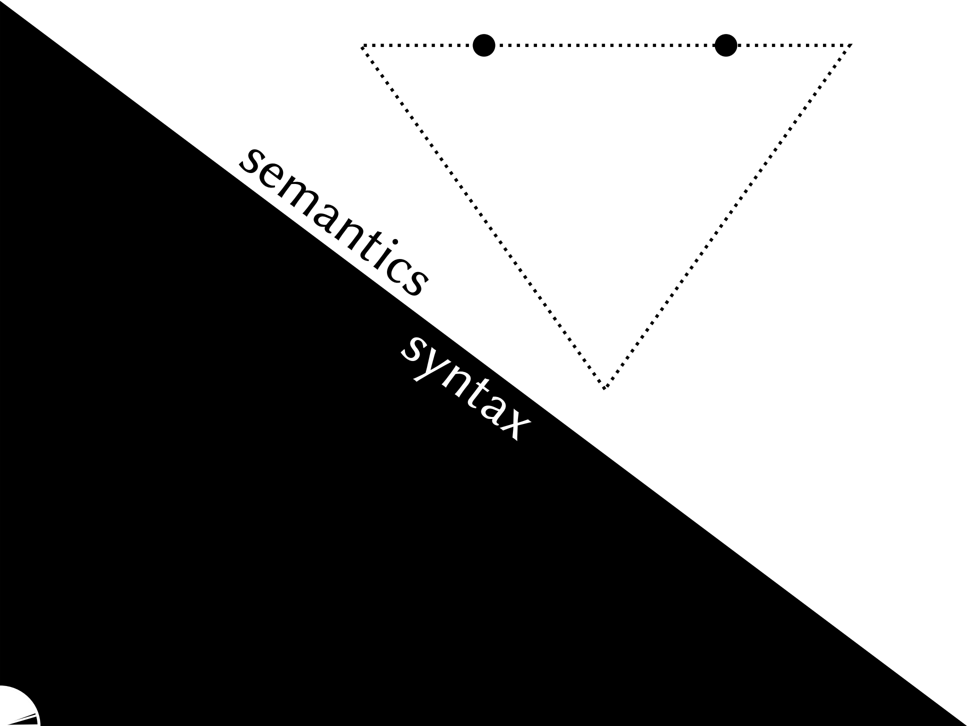


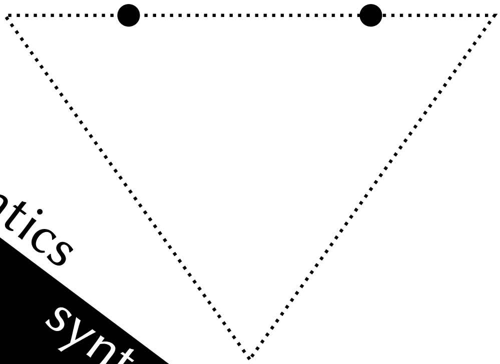







semantics


semantics
syntax



semantics

syntax

$$\frac{A \vee B}{\neg\neg A, \neg\neg B}$$

$$A \vdash B$$

$$\frac{A \vdash B}{A \mathrel{|\!\!\sim} B}$$

Admissible Approximation

$$\underline{A} \vdash B \text{ iff } A \mathrel{|\!\!\sim} B$$

If admissible approximations exists,
and if $A \vdash_{\mathrm{R}} \underline{A}$
then $\mathrel{|\!\sim} \subseteq \vdash_{\mathrm{R}}$.

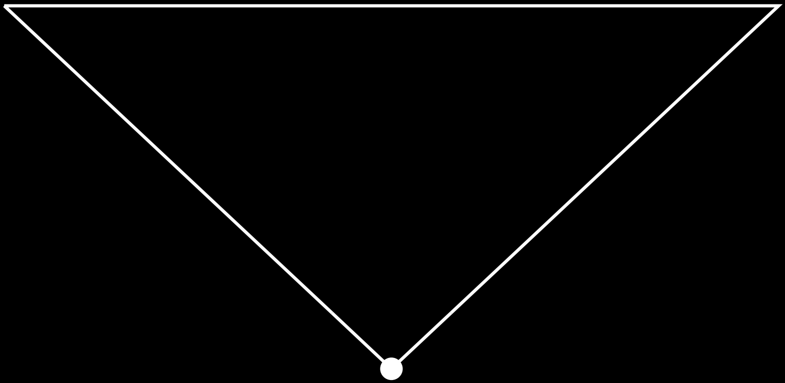
If admissible approximations exists,
and if $A \vdash_{\mathrm{R}} \underline{A}$ and $\mathrm{R} \subseteq \mathrel{|\!\sim}$
then $\mathrel{|\!\sim} = \vdash_{\mathrm{R}}$.

A is projective when
$\vdash \sigma A$ and $A \vdash \sigma B \equiv B$
for some σ.

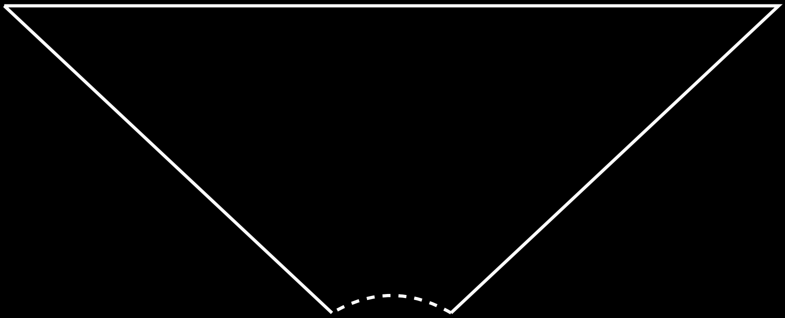
A is projective when
$\vdash \sigma A$ and $A \vdash \sigma B \equiv B$
for some σ.

$$\underline{A} = A$$

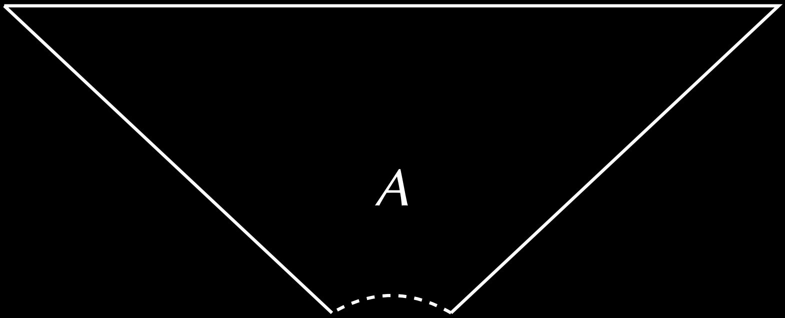
Ghilardi (1999)



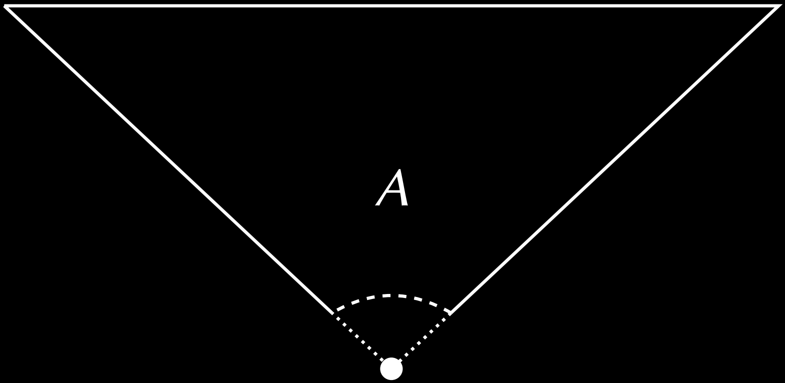
Ghilardi (1999)



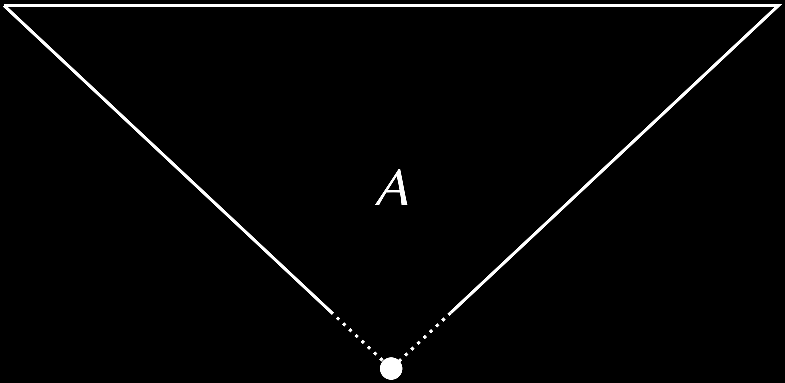
Ghilardi (1999)



Ghilardi (1999)



Ghilardi (1999)



Iemhoff (2001b)

A formula is IPC-projective iff it admits DP and V

Goudsmit and Iemhoff (2014)

A formula is T_n -projective iff
it admits DP and V_n
for $n \geq 2$

Visser Rules

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\bigvee \{ \quad (\bigvee \Delta \to A) \to C \ \mid C \in \Delta \}}$$

Skura (1992)

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\{\neg\neg((\bigvee \Delta \to A) \to C) \mid C \in \Delta\}}$$

A formula is BD_2-projective iff
it admits S

To each A there is set Γ of BD_2-projectives with

$$A \vdash_S \bigvee \Gamma \text{ and } \bigvee \Gamma \vdash A$$

To each A there is set Γ of BD_2-projectives with

$$A \vdash_S \bigvee \Gamma \text{ and } \bigvee \Gamma \vdash A$$

which shows $\underline{A} = \bigvee \Gamma$.

Goudsmit (2013):

S axiomatises admissibility of BD_2

References I

- Citkin, Alexander (1979). “О Проверке Допустимости не́которых Правил Интуиционистской Логике”. Russian. In: *V-th All-Union Conference in Mathematical Logic*. English translation of title: On verification of admissibility of some rules of intuitionistic logic. Novosibirsk, p. 162.
- Gabbay, Dov M. and Dick Herman Jacobus de Jongh (1974). “A Sequence of Decidable Finitely Axiomatizable Intermediate Logics with the Disjunction Property”. In: *The Journal of Symbolic Logic* 39.1, pp. 67–78. ISSN: 00224812. JSTOR: 2272344.
- Ghilardi, Silvio (1999). “Unification in Intuitionistic Logic”. In: *The Journal of Symbolic Logic* 64.2, pp. 859–880. ISSN: 00224812. JSTOR: 2586506.
- Goudsmit, Jeroen P. (2013). “The Admissible Rules of BD_2 and GSc”. In: *Logic Group Preprint Series* 313, pp. 1–24. URL: http://phil.uu.nl/preprints/lgps/number/313.

References II

Goudsmit, Jeroen P. and Rosalie Iemhoff (2014). "On unification and admissible rules in Gabbay-de Jongh logics". In: *Annals of Pure and Applied Logic* 165.2, pp. 652–672. ISSN: 0168-0072. DOI: 10.1016/j.apal.2013.09.003. URL: http://phil.uu.nl/preprints/lgps/number/297.

Harrop, Ronald (1960). "Concerning Formulas of the Types $A \to A \vee B, A \to (\exists x)B(x)$ in Intuitionistic Formal Systems". In: *The Journal of Symbolic Logic* 25.1, pp. 27–32. ISSN: 00224812. JSTOR: 2964334.

Iemhoff, Rosalie (2001a). "A(nother) characterization of intuitionistic propositional logic". In: *Annals of Pure and Applied Logic* 113.1-3. First St. Petersburg Conference on Days of Logic and Computability, pp. 161–173. ISSN: 0168-0072. DOI: 10.1016/S0168-0072(01)00056-2.

References III

- Iemhoff, Rosalie (2001b). “On the Admissible Rules of Intuitionistic Propositional Logic”. In: *The Journal of Symbolic Logic* 66.1, pp. 281–294. ISSN: 00224812. JSTOR: 2694922.
- Kreisel, Georg and Hilary Whitehall Putnam (1957). “Eine Unableitbarkeitsbeweismethode für den Intuitionistischen Aussagenkalkül”. In: *Archiv für mathematische Logik und Grundlagenforschung* 3 (3-4), pp. 74–78. ISSN: 0003-9268. DOI: 10.1007/BF01988049.
- Łukasiewicz, Jan (1952). “On the intuitionistic theory of deduction”. In: *Indagationes Mathematicae* 14, pp. 202–212.
- Maksimova, Larisa L. (1986). “On Maximal Intermediate Logics with the Disjunction Property”. In: *Studia Logica: An International Journal for Symbolic Logic* 45.1, pp. 69–75. ISSN: 00393215. JSTOR: 20015248.

References IV

- Mints, Grigori E. (1976). "Derivability of admissible rules". In: *Journal of Mathematical Sciences* 6 (4), pp. 417–421. ISSN: 1072-3374. DOI: 10.1007/BF01084082.
- Rozière, Paul (1992). "Règles admissibles en calcul propositionnel intuitionniste". PhD thesis. Université de Paris VII.
- Scott, Dana (1957). "Completeness Proofs for the Intuitionistic Sentential Calculus". In: *Summaries of talks presented at the summer institute for symbolic logic.* Second Edition 25 July 1960. Communications Research Division, Institute for Defence Analyses, pp. 231–241.
- Skura, Tomasz F. (1989). "A complete syntactical characterization of the intuitionistic logic". In: *Reports on Mathematical Logic* 23, pp. 75–80.
- – (1992). "Refutation Calculi for Certain Intermediate Propositional Logics". In: *Notre Dame Journal of Formal Logic* 33.4, pp. 552–560. DOI: 10.1305/ndjfl/1093634486.

References V

Visser, Albert (1984). "Evaluation, provably deductive equivalence in Heyting's arithmetic of substitution instances of propositional formulas". In: *Logic Group Preprint Series* 4. URL: http://phil.uu.nl/preprints/lgps/number/4.