The Admissible Rules of BD_2

Jeroen Goudsmit
Utrecht University
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Disjunction Property

$$\frac{A \vee B \text{ derivable}}{A \text{ derivable or } B \text{ derivable}}$$

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

syntax

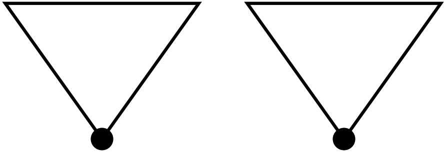
$$\frac{\vdash A \vee B}{\vdash A \text{ or} \vdash B}$$

semantics

syntax

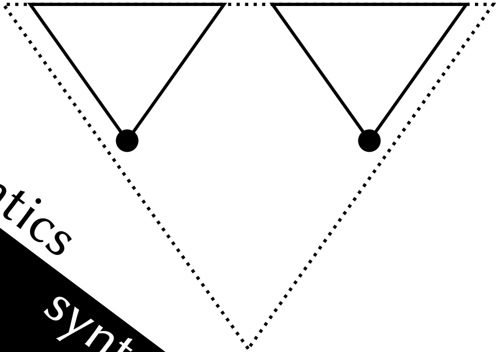
$$\frac{\vdash A \lor B}{\vdash A \text{ or } \vdash B}$$

semantics



syntax

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$



semantics

syntax

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

Logic of Depth n

$$\mathrm{bd}_0 = \bot$$
$$\mathrm{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathrm{bd}_n).$$

Logic of Depth n

$$\begin{aligned} \mathrm{bd}_0 &= \bot \\ \mathrm{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathrm{bd}_n). \end{aligned}$$

$$\mathrm{BD}_n = \mathrm{IPC} + \mathrm{bd}_n$$

Overview

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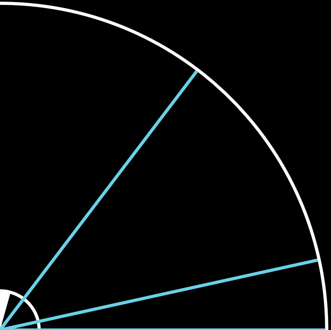
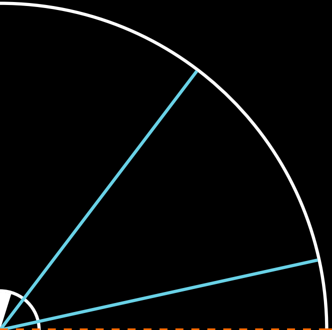
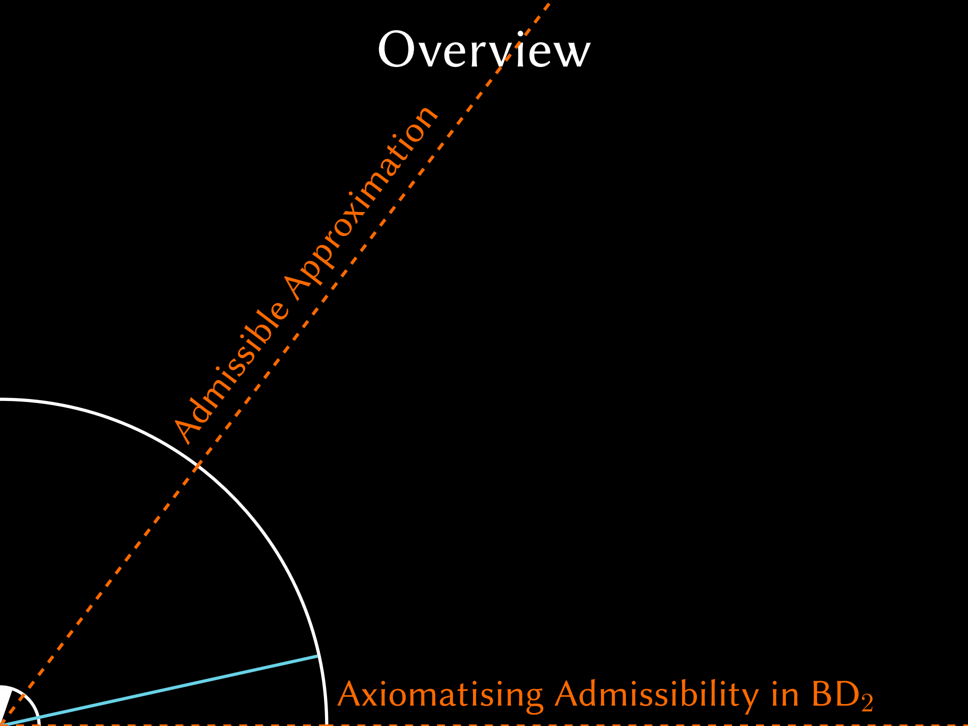


Overview

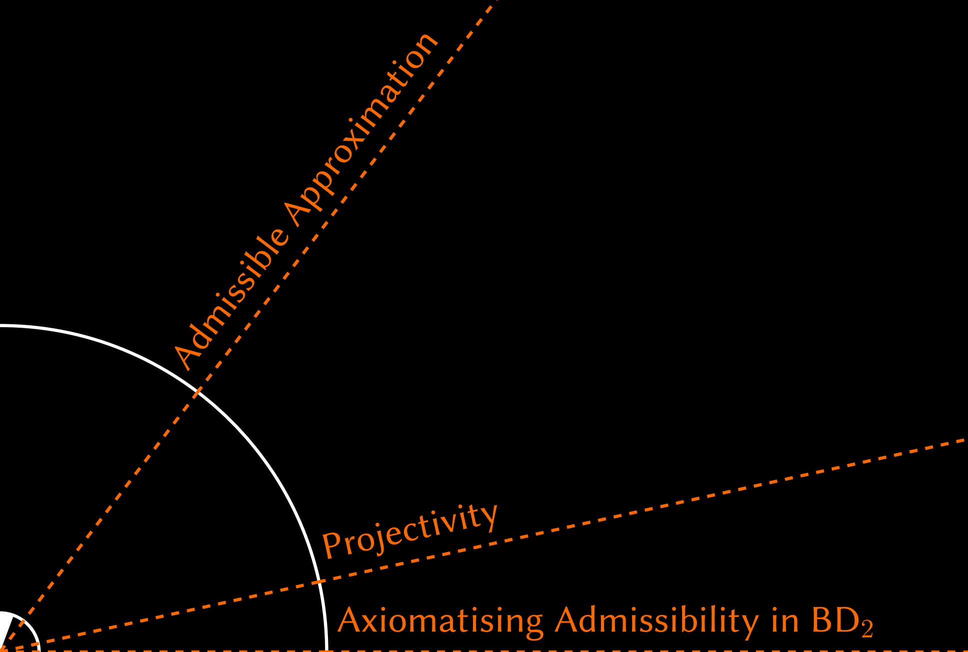


Axiomatising Admissibility in BD_2

Overview


Admissible Approximation
Axiomatising Admissibility in BD_2

Overview



Admissible Approximation
Projectivity
Axiomatising Admissibility in BD_2

A / $\triangle$ admissible

σA is derivable

$$A \;/\; \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

σA is derivable

$$A \mathrel{|\!\sim} \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

$$\frac{\neg C \to A \lor B}{(\neg C \to A) \lor (\neg C \to B)}$$

$$\frac{\neg C \to A \lor B}{\{\ \neg C \to A, \quad \neg C \to B\ \}}$$

¬¬ Disjunction Property

$$\frac{\bigvee \Delta}{\{\neg\neg C \mid C \in \Delta\}}$$

An axiomatisation of admissibility is a set of rules R with

$$\vdash_{R} = \mathrel{|\!\!\sim}$$

$$A \vdash B$$

$$\frac{A \vdash B}{A \mathrel{|\!\sim} B}$$

Admissible Approximation

$$\underline{A} \vdash B \text{ iff } A \mathrel{|\!\!\sim} B$$

If admissible approximations exists,
and if $A \vdash_{\mathrm{R}} \underline{A}$
then $\mid\!\sim\ \subseteq\ \vdash_{\mathrm{R}}$.

If admissible approximations exists,
and if $A \vdash_{\mathrm{R}} \underline{A}$ and $\mathrm{R} \subseteq \mathrel{|\!\!\sim}$
then $\mathrel{|\!\!\sim} = \vdash_{\mathrm{R}}$.

Visser Rules

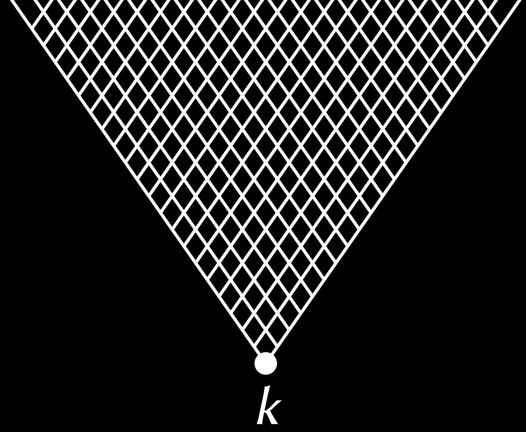
$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\bigvee \left\{ (\bigvee \Delta \to A) \to C \;\middle|\; C \in \Delta \right\}}$$

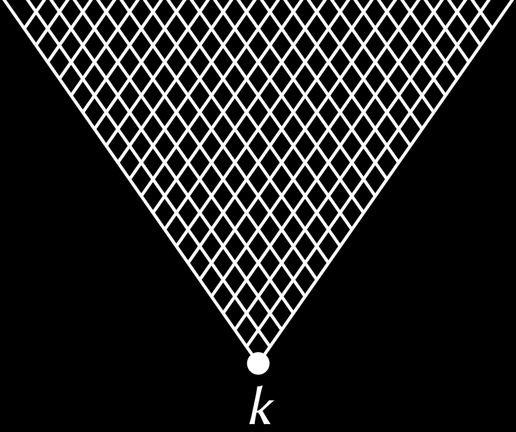
Jankov–de Jongh formulae

In suitable models have

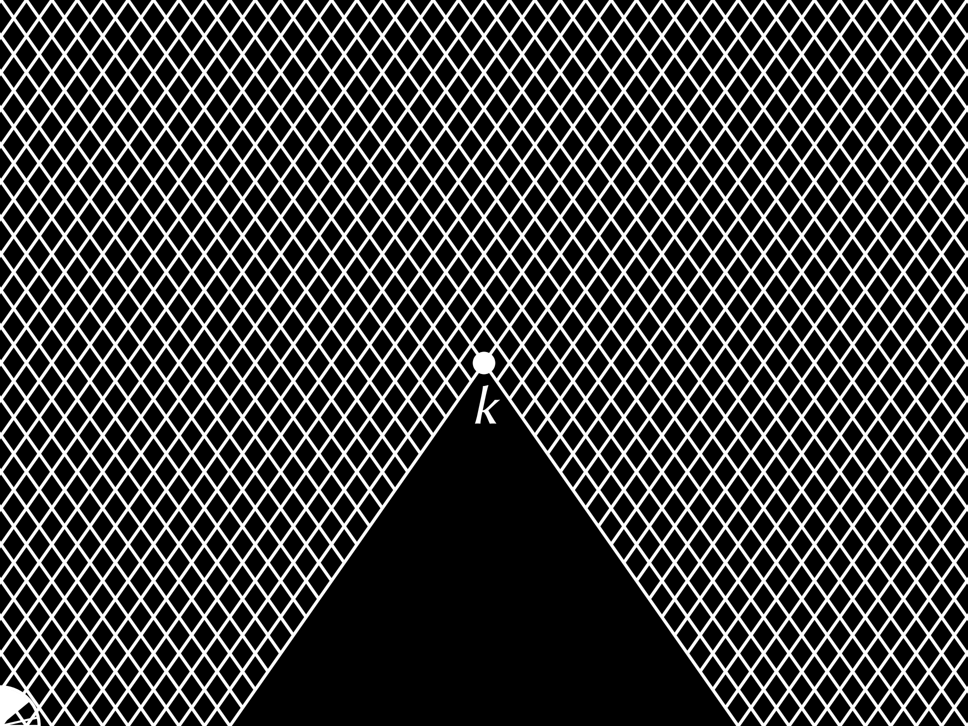
$$l \Vdash \mathrm{up}\, k \quad \text{iff} \quad k \leq l$$
$$l \Vdash \mathrm{nd}\, k \quad \text{iff} \quad l \not\leq k$$

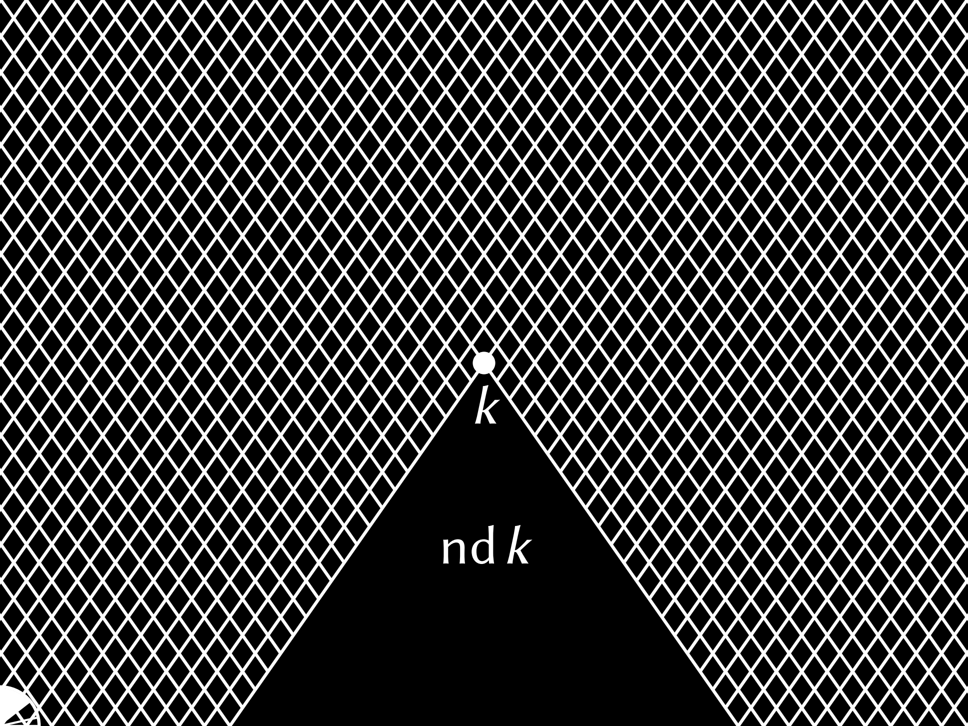

k

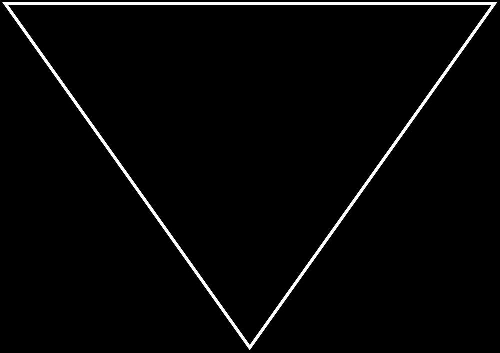

k

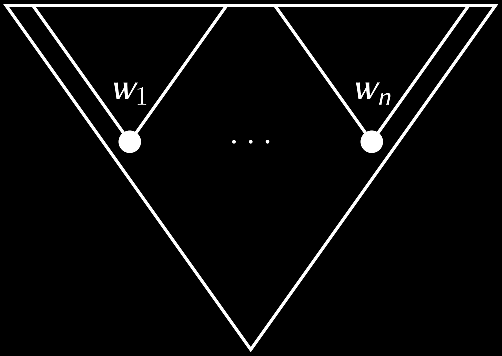


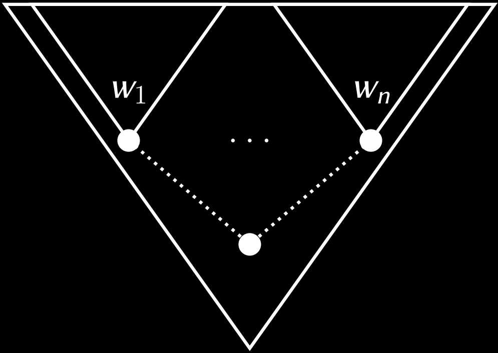
up k

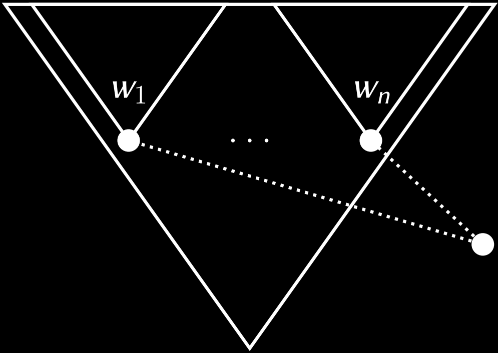

k


k
nd k

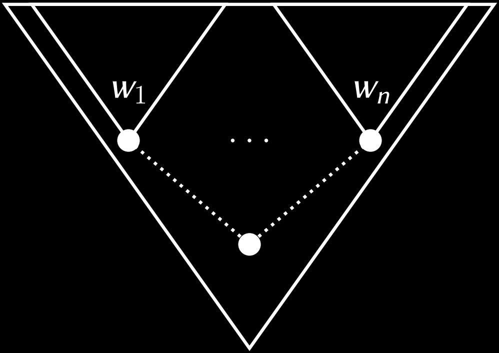


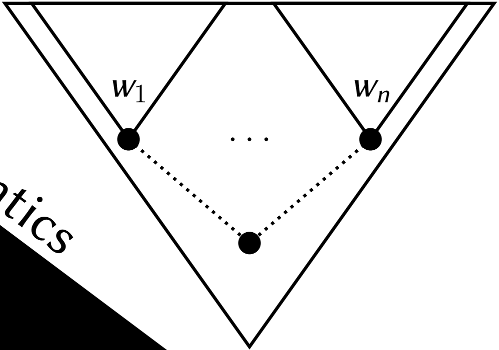

w_1
w_n


w_1
w_n
. . .

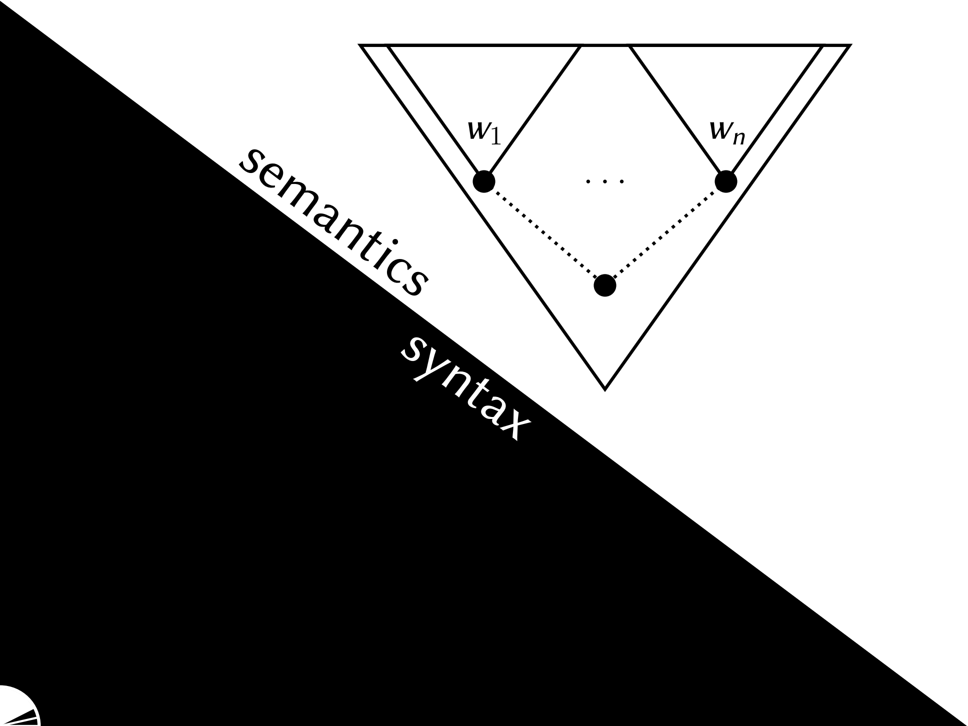


w_1
. . .
w_n


w_1
w_n
$\cdots$



semantics



semantics
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w_1
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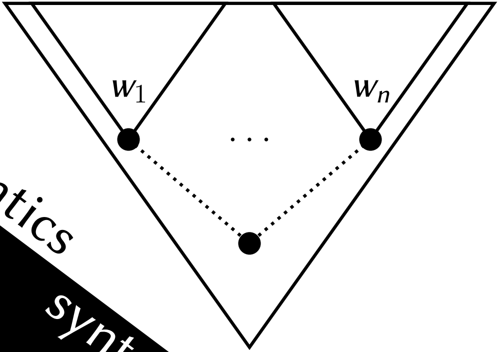
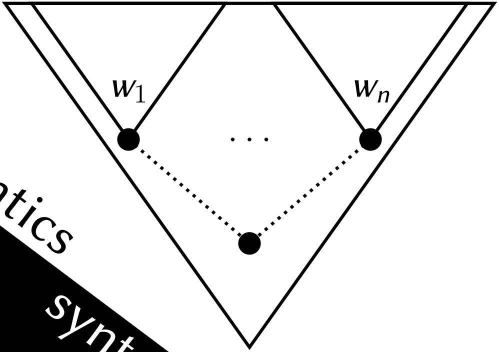


semantics

syntax

$$\frac{\left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \bigvee_{i=1}^{n} \mathsf{nd}\, w_i}{\bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} \mathsf{nd}\, w_i \to \bigvee_{i=1}^{n} \mathsf{up}\, w_i\right) \to \mathsf{nd}\, w_j}$$



semantics

syntax

$$\frac{\left(\bigvee \Delta \to A\right) \to \bigvee \Delta}{\bigvee_{C \in \Delta} \left(\bigvee \Delta \to A\right) \to C}$$

A is projective when

$\vdash \sigma A$ and $A \vdash \sigma B \equiv B$

for some σ.

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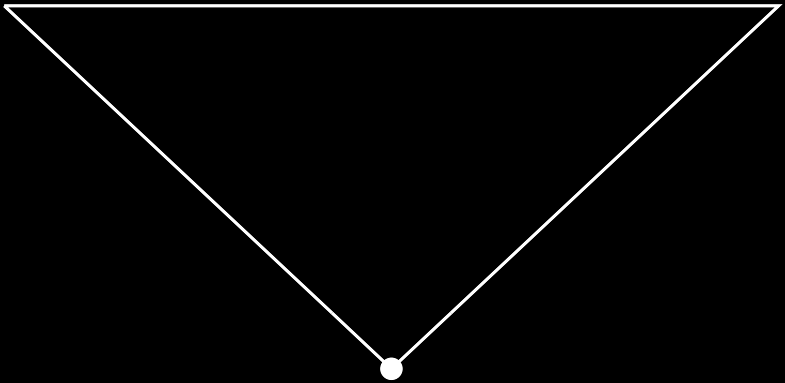
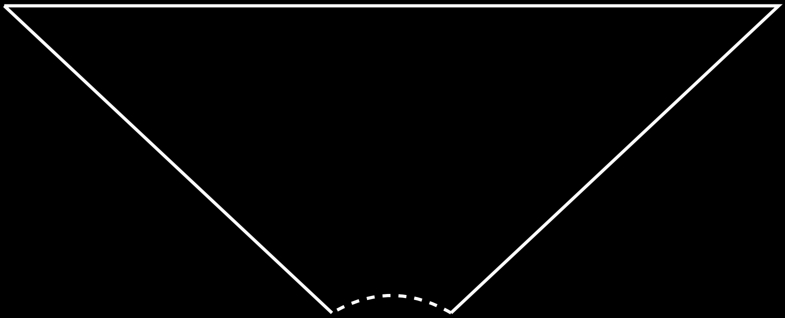
$$\underline{A} = A$$

Ghilardi (1999)



Ghilardi (1999)



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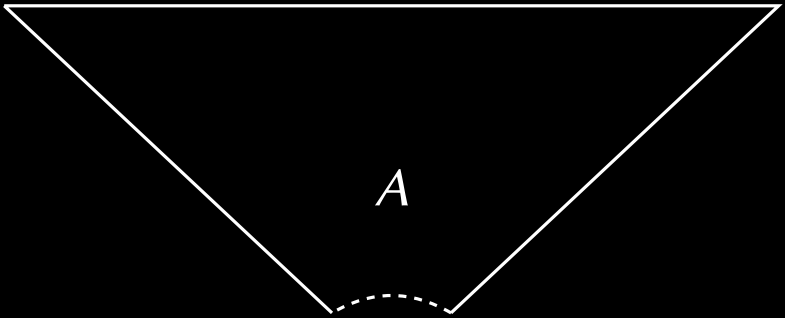

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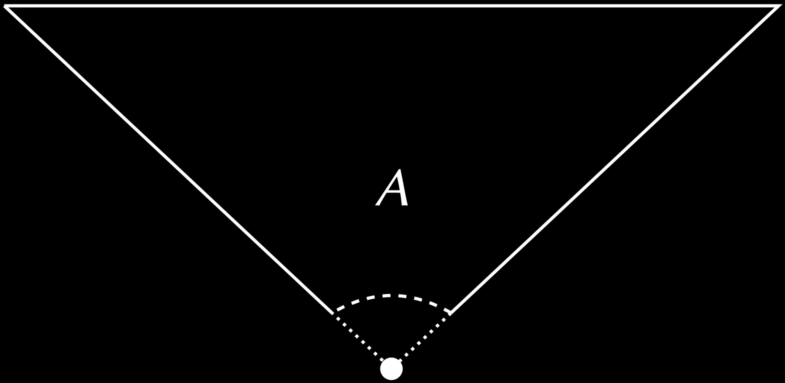

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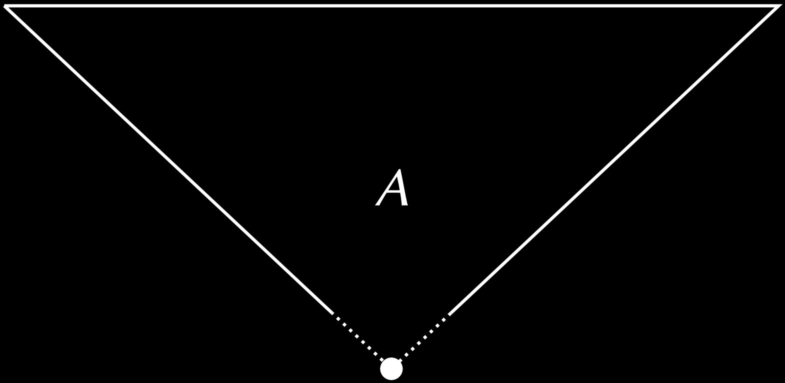

Iemhoff (2001)

A formula is IPC-projective iff
it admits DP and V

Goudsmit and Iemhoff (2012)

A formula is T_n -projective iff
it admits DP and V_n
for $n \geq 2$

Visser Rules

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\bigvee \{ \quad (\bigvee \Delta \to A) \to C \ \mid C \in \Delta \}}$$

Skura (1992)

$$\frac{(\bigvee \Delta \to A) \to \bigvee \Delta}{\{\neg\neg((\bigvee \Delta \to A) \to C) \mid C \in \Delta\}}$$

A formula is BD_2-projective iff it admits S

To each A there is set Γ of BD_2-projectives with

$$A \vdash_S \bigvee \Gamma \text{ and } \bigvee \Gamma \vdash A$$

To each A there is set Γ of BD_2-projectives with

$$A \vdash_S \bigvee \Gamma \text{ and } \bigvee \Gamma \vdash A$$

which shows $\underline{A} = \bigvee \Gamma$.

Goudsmit (2013):

S axiomatises admissibility of BD_2

References I

- Ghilardi, Silvio (1999). "Unification in Intuitionistic Logic". In: *The Journal of Symbolic Logic* 64.2, pp. 859–880. ISSN: 00224812. JSTOR: 2586506.
- Goudsmit, Jeroen P. (2013). "The Admissible Rules of BD_2 and GSc". In: *Logic Group Preprint Series* 313, pp. 1–23. URL: http://phil.uu.nl/preprints/lgps/number/313.
- Goudsmit, Jeroen P. and Rosalie Iemhoff (2012). "On unification and admissible rules in Gabbay-de Jongh logics". In: *Logic Group Preprint Series* 297, pp. 1–18. ISSN: 0929-0710. URL: http://phil.uu.nl/preprints/lgps/number/297.
- Iemhoff, Rosalie (2001). "On the Admissible Rules of Intuitionistic Propositional Logic". In: *The Journal of Symbolic Logic* 66.1, pp. 281–294. ISSN: 00224812. JSTOR: 2694922.
- Skura, Tomasz F. (1992). "Refutation Calculi for Certain Intermediate Propositional Logics". In: *Notre Dame Journal of Formal Logic* 33.4, pp. 552–560. DOI: 10.1305/ndjfl/1093634486.